Moment Determinacy of Products of Non-identically Distributed Random Variables

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Abstract

We show first that there are intrinsic relationships among different conditions, old and recent, which lead to some general statements in both the Stieltjes and the Hamburger moment problems. Then we describe checkable conditions and prove new results about the moment (in)determinacy for products of independent and non-identically distributed random variables. We treat all three cases when the random variables are nonnegative (Stieltjes case), when they take values in the whole real line (Hamburger case), and the mixed case. As an illustration we characterize the moment determinacy of products of random variables whose distributions are generalized gamma or double generalized gamma all with distinct shape parameters. Among other corollaries, the product of two independent random variables, one exponential and one inverse Gaussian, is moment determinate, while the product is moment indeterminate for the cases: one exponential and one normal, one chi-square and one normal, and one inverse Gaussian and one normal.

Keywords: Cramér's condition, Carleman's condition, Krein's condition, Hardy's condition.

1. Introduction

There is a long standing interest in studying products of random variables; see, e.g., Lévy (1959), Lomnicki (1967), Springer (1979), Cline & Samorodnitsky (1994), and Shimura (2000). The reasons are two-fold. On one hand, to deal with products leads to non-trivial, difficult and challenging theoretical problems requiring to use diverse ideas and techniques. Let us mention just a few sources: Galambos & Simonelli (2004), Berg (2005), and Bondesson (2013). On the other hand, products of random variables are naturally involved in stochastic modelling of complex random phenomena in areas such as statistical physics, quantum theory, communication theory and financial modelling; see, e.g., Carmona & Molchanov (1993), Davis & Mikosch (2009), Hashorva et al. (2010), de Abreu (2010), and Chen et al. (2012).

In general, it is rare to find explicit closed-form expressions for the densities or the distributions of products of random variables with different distributions. It is, however, possible to study successfully the moment problem for products of independent random variables; see, e.g., Lin & Stoyanov (2014), and Stoyanov et al. (2014). Answers about the moment (in)determinacy can be found if requiring only information about the asymptotics of the moments or about the tails of the densities or of their distributions.

All random variables considered in this paper are defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and we denote by $\mathbf{E}[X]$ the expected value of the random variable X. A basic assumption is that the random variables we deal with have finite moments of all positive orders, i.e. $\mathbf{E}[|X|^k] < \infty, \ k = 1, 2, \ldots$. We write $X \sim F$ to mean that X is a random variable whose distribution function is F and denote its kth order moment by $m_k = \mathbf{E}[X^k]$. We say that X or F is *moment determinate* (M-det) if F is the only distribution having the moment sequence $\{m_k\}_{k=1}^{\infty}$; otherwise, we say that X or F is *moment indeterminate* (M-indet). We use traditional notions, notations and terms such as Cramér's condition, Carleman's condition, Krein's condition, and Hardy's condition (see, e.g., Pakes, 2001, Lin & Stoyanov, 2014, and Stoyanov et al., 2014). We use $\Gamma(\cdot)$ for the Euler-gamma function, $R = (-\infty, \infty)$ for the set of all real numbers, $R_+ = [0, \infty)$ for the nonnegative numbers, the symbol $\mathcal{O}(\cdot)$ with its usual meaning in asymptotic analysis and the abbreviation i.i.d. for independent and identically distributed (random variables).

In Section 2 we describe useful intrinsic relationships among different old and recent conditions involved in the Stieltjes and/or the Hamburger moment problems. Then we present some new results under conditions which

are relatively easy to check. In Section 3 we deal with the moment determinacy of products of independent nonnegative random variables with different distributions, while in Section 4 we consider products of random variables with values in R. Finally, in Section 5, we treat the mixed case: products of both types of random variables, nonnegative ones and real ones, the latter with values in R.

The results presented in this paper extend some previous results for products of i.i.d. random variables. Here we need a more refined analysis of the densities of products than in the i.i.d. case. As an illustration we characterize the moment (in)determinacy of products of random variables whose distributions are generalized gamma or double generalized gamma all with distinct shape parameters. We have derived several corollaries involving popular distributions widely used in theoretical studies and applications. Let us list a few: (i) the product of two independent random variables, one exponential and one inverse Gaussian, is M-det; (ii) the product of independent exponential and normal random variables is M-indet; (iii) the product of independent chi-square and normal random variables is M-indet; and (iv) the product of independent inverse Gaussian and normal random variables is M-indet.

2. Some General Results

In this section we present two lemmas, each containing workable conditions which, more or less, are available in the literature. Some of these conditions are old, while others are recent. We describe intrinsic relationships among these conditions and use them to obtain new results; see Theorems 1–4.

Our findings in this section can be considered as a useful complement to the classical criteria of Cramér, Carleman, Krein, Hardy and their converses, so that all these taken together make more clear, and possibly complete, the picture of what is in our hands when discussing the determinacy of distributions in terms of their moments.

2.1. Stieltjes Case

We present the first important lemma.

Lemma 1. *Let $0 \leq X \sim F$. Then the following statements are equivalent:*
(i) $m_k = \mathcal{O}(k^{2k})$ as $k \to \infty$.
(ii) $\limsup_{k \to \infty} \frac{1}{k} m_k^{1/(2k)} < \infty$.
(iii) $m_k \leq c_0^k (2k)!, \ k = 1, 2, \ldots,$ for some constant $c_0 > 0$.
(iv) X satisfies Hardy's condition, namely, $\mathbf{E}[\exp(c\sqrt{X})] < \infty$ for some constant $c > 0$.

Theorem 1. *Let $0 \leq X \sim F$ with moments growing as follows: $m_k = \mathcal{O}(k^{ak})$ as $k \to \infty$ for some constant $a \in (0, 2]$. Then the following statements hold:*
(i) X satisfies Hardy's condition and hence X is M-det.
(ii) The boundary value $a = 2$ is the best possible for X to be M-det. In fact, there is an M-indet random variable $\tilde{X} \geq 0$ such that $\mathbf{E}[\tilde{X}^k] = \mathcal{O}(k^{ak})$ as $k \to \infty$ for all $a > 2$.

Remark 1. For $0 \leq X \sim F$, let us compare the following two moment conditions: (a) $m_k = \mathcal{O}(k^{2k})$ as $k \to \infty$, and (b) $m_{k+1}/m_k = \mathcal{O}((k+1)^2)$ as $k \to \infty$. Here (a) is the condition in Theorem 1, while condition (b) was introduced and used in the recent paper Lin & Stoyanov (2014). Both conditions are checkable and each of them guarantees the moment determinacy of F. Just to mention that condition (b) implies condition (a) by referring to Theorem 3 in Lin & Stoyanov (2014), while the converse may not be true in general.

The next result, Theorem 2 below, is the converse to Theorem 1, and deals with the moment indeterminacy of nonnegative random variables. We need first one condition which is used a few times in the sequel.

Condition L: Suppose, in the Stieltjes case, that $f(x)$, $x \in R_+$, is a density function such that for some fixed $x_0 > 0$, f is strictly positive and differentiable for $x > x_0$ and

$$ L_f(x) := -\frac{xf'(x)}{f(x)} \nearrow \infty \quad \text{as} \ \ x_0 < x \to \infty. $$

In the Hamburger case we require the density $f(x)$, $x \in R$, to be symmetric.

This condition plays a significant rôle in moment problems for absolutely continuous probability distributions. It was explicitly introduced and efficiently used for the first time in Lin (1997) and later used by several authors naming it as 'Lin's condition'. This condition is involved in some of our results to follow.

Theorem 2. *Let $0 \leq X \sim F$ and its moment sequence $\{m_k, \ k = 1, 2, \ldots\}$ grow 'fast' in the sense that $m_k \geq c\, k^{(2+\varepsilon)k}, \ k = 1, 2, \ldots,$ for some constants $c > 0$ and $\varepsilon > 0$. Assume further that X has a density function f which satisfies the above Condition L. Then X is M-indet.*

Remark 2. To provide one application of Theorem 2, let us consider, for example, the random variable $X = \xi^{2(1+\varepsilon)}$, where $\varepsilon > 0$ and $\xi \sim Exp(1)$, the standard exponential distribution. On one hand, we can use the Krein criterion and show that X is M-indet. On the other hand, X satisfies the moment condition in Theorem 2. And here is the point: instead of applying Krein's condition, we can prove the moment indeterminacy of X by checking that its density f satisfies Condition L. In general, we follow the approach which is easier.

2.2. Hamburger Case

We start with Lemma 2 establishing the equivalence of different type of conditions involved to decide whether a distribution on the whole real line R is M-det. Then we present some new results. Theorem 3 below is a slight modification, in a new light, of a result in Bisgaard & Sasvari (2000), p. 92, while Theorem 4 is the converse to Theorem 3.

Lemma 2. *Let X be a random variable taking values in R. Then the following statements are equivalent:*
(i) $m_{2k} = \mathcal{O}((2k)^{2k})$ as $k \to \infty$.
(ii) $\limsup_{k\to\infty} \frac{1}{2k} m_{2k}^{1/(2k)} < \infty$.
(iii) $m_{2k} \le c_0^k (2k)!$, $k = 1, 2, \ldots$, for some constant $c_0 > 0$.
(iv) X satisfies Cramér's condition: its moment generating function exists.

Theorem 3. *Let $X \sim F$, where F has an unbounded support $supp(F) \subset R$ and its moments satisfy the condition: $m_{2k} = \mathcal{O}((2k)^{2ak})$ as $k \to \infty$ for some constant $a \in (0, 1]$. Then the following statements hold:*
(i) X satisfies Cramér's condition and hence is M-det.
(ii) The boundary value $a = 1$ is the best possible for X to be M-det. In fact, there is an M-indet random variable $\tilde{X}$ such that $\mathbf{E}[\tilde{X}^{2k}] = \mathcal{O}((2k)^{2ak})$ as $k \to \infty$ for all $a > 1$.

Remark 3. Suppose $X \sim F$ with F having unbounded support, $supp(F) \subset R$. We want to compare the following two moment conditions: (a) $m_{2k} = \mathcal{O}((2k)^{2k})$ as $k \to \infty$, and (b) $m_{2(k+1)}/m_{2k} = \mathcal{O}((k+1)^2)$ as $k \to \infty$. Here (a) is the growth of the moments condition in Theorem 3, while condition (b) was introduced and successfully exploited in the recent work Stoyanov et al. (2014). Both conditions are checkable and each of them guarantees the moment determinacy of X and F. Let us mention that condition (b) implies condition (a) by referring to Theorem 2 in Stoyanov et al. (2014), while the converse may not in general be true.

Theorem 4. *Suppose that the moments of $X \sim F$ grow 'fast' in the sense that $m_{2k} \ge c(2k)^{2(1+\varepsilon)k}$, $k = 1, 2, \ldots$, for some positive constants c and ε. Assume further that X has a density function f which is symmetric about zero and satisfies the above Condition L. Then X is M-indet.*

Remark 4. For example, instead of applying Krein's condition, we can use Theorem 4 to prove the moment indeterminacy of $X \sim F$ whose density is the symmetrization of that of $\xi^{1+\varepsilon}$, where $\varepsilon > 0$ and $\xi \sim Exp(1)$.

3. Products of Nonnegative Random Variables

We start with two results describing relatively simple conditions on the random variables $\xi_1, \ldots, \xi_n$ in order to guarantee that their product is M-det.

Theorem 5. *Suppose that the moments $m_{i,k} = \mathbf{E}[\xi_i^k]$, $i = 1, \ldots, n$, of the independent nonnegative random variables $\xi_1, \ldots, \xi_n$ satisfy the conditions:*

$$m_{i,k} = \mathcal{O}(k^{a_i k}) \quad as \quad k \to \infty, \quad for \quad i = 1, \ldots, n,$$

where $a_1, \ldots, a_n$ are positive constants. If the constants $a_1, \ldots, a_n$ are such that $a_1 + \cdots + a_n \le 2$, then the product $Z_n = \xi_1 \cdots \xi_n$ is M-det.

Theorem 6. *Suppose that the growth rates $r_1, \ldots, r_n$ of the moments of the independent nonnegative random variables $\xi_1, \ldots, \xi_n$ satisfy*

$$\frac{m_{1,k+1}}{m_{1,k}} = \mathcal{O}((k+1)^{r_1}), \quad \ldots, \quad \frac{m_{n,k+1}}{m_{n,k}} = \mathcal{O}((k+1)^{r_n}) \quad as \quad k \to \infty,$$

where $m_{i,k} = \mathbf{E}[\xi_i^k]$, $i = 1, \ldots, n$, $k = 1, 2, \ldots$. If the rates $r_1, \ldots, r_n$ are such that $r_1 + \cdots + r_n \le 2$, then the product $Z_n = \xi_1 \cdots \xi_n$ is M-det.

Let us provide now conditions under which the product Z_n becomes M-indet.

Theorem 7. *Consider n independent nonnegative random variables, $\xi_i \sim F_i$, $i = 1, 2, \ldots, n$, where $n \ge 2$. Suppose that each F_i has a density $f_i > 0$ on $(0, \infty)$ and that the following conditions are satisfied:*

(i) At least one of the densities $f_1(x), \ldots, f_n(x)$ is decreasing in $x \geq x_0$, where $x_0 \geq 1$ is a constant.
(ii) For each $i = 1, 2, \ldots, n$, there exists a constant $A_i > 0$ such that the density f_i and the tail function $\overline{F_i} = 1 - F_i$ satisfy the relation

$$f_i(x)/\overline{F_i}(x) \geq A_i/x \quad for \quad x \geq x_0, \tag{1}$$

and there exist constants $B_i > 0$, $\alpha_i > 0$, $\beta_i > 0$ and real γ_i such that

$$\overline{F_i}(x) \geq B_i x^{\gamma_i} \exp(-\alpha_i x^{\beta_i}) \quad for \quad x \geq x_0. \tag{2}$$

If, in addition to (i) and (ii), the parameters $\beta_1, \ldots, \beta_n$ are such that $\sum_{i=1}^n 1/\beta_i > 2$, then the product $Z_n = \xi_1 \xi_2 \cdots \xi_n$ is M-indet.

To prove Theorem 7, we need the following lemma.

Lemma 3. *Let F be a distribution on R such that (i) it has density f on the subset $[a, ra]$, where $a > 0$ and $r > 1$, and (ii) for some constant $A > 0$, $f(x)/\overline{F}(x) \geq A/x$ on $[a, ra]$. Then*

$$\int_a^{ra} \frac{f(x)}{x} dx \geq \left(1 - \frac{1}{r}\right) \frac{A}{1+A} \frac{\overline{F}(a)}{a}.$$

Example 1. For illustration of how to use Theorem 7, consider the class of generalized gamma distributions. We use the notation $\xi \sim GG(\alpha, \beta, \gamma)$ if the density function of the random variable ξ is of the form

$$f(x) = cx^{\gamma-1} \exp(-\alpha x^\beta), \quad x \geq 0. \tag{3}$$

Here $\alpha, \beta, \gamma > 0$, $f(0) = 0$ if $\gamma \neq 1$, and $c = \beta \alpha^{\gamma/\beta}/\Gamma(\gamma/\beta)$ is the norming constant. We have the following statement (see also Theorem 8.4 in Pakes, 2014, for a more general result with different proof).

Corollary 1. *Suppose $\xi_1, \ldots, \xi_n$ are n independent random variables such that $\xi_i \sim GG(\alpha_i, \beta_i, \gamma_i)$, $i = 1, \ldots, n$, and let $Z_n = \xi_1 \cdots \xi_n$. Then Z_n is M-det if and only if $\sum_{i=1}^n 1/\beta_i \leq 2$.*

Example 2. Consider the class of inverse Gaussian distributions. We say that $X \sim IG(\mu, \lambda)$ if the density of X is of the form

$$f(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right], \quad x > 0, \tag{4}$$

where $\mu, \lambda > 0$ and $f(0) = 0$. If $X \sim IG(\mu, \lambda)$, then it has a moment generating function. This in turn implies that the power $Y = X^2$ satisfies Hardy's condition and hence is M-det. Actually, we have that for real r, X^r is M-det if and only if $|r| \leq 2$ (see Stoyanov, 1999). If ξ_1 and ξ_2 are two i.i.d. random variables with density (4), then the product $Z = \xi_1 \xi_2$ is also M-det due to Proposition 1(iii) in Lin & Stoyanov (2014). The next result is for products of non-identically distributed random variables.

Corollary 2. *Let $\xi_1 \sim IG(\mu_1, \lambda_1)$, $\xi_2 \sim IG(\mu_2, \lambda_2)$ and $\eta \sim Exp(1)$ be three independent random variables. Then the following statements hold:*
(i) $Z = \xi_1 \eta$ is M-det.
(ii) $Z = \xi_1 \xi_2$ is M-det.
(iii) $Z = \xi_1 \xi_2 \eta$ is M-indet.

4. Products of Random Variables in R

We start with two results describing relatively simple conditions on the random variables $\xi_1, \ldots, \xi_n$ in order to guarantee that their product is M-det. The results are similar to the above Theorems 5 and 6, however we remember that here we deal with the Hamburger case, so we work with the even order moments.

Theorem 8. *Suppose that the even order moments $m_{i,2k} = \mathbf{E}[\xi_i^{2k}]$, $i = 1, \ldots, n$, of the independent random variables $\xi_1, \ldots, \xi_n$ satisfy the conditions:*

$$m_{i,2k} = \mathcal{O}((2k)^{2a_i k}) \quad as \quad k \to \infty, \quad for \quad i = 1, \ldots, n,$$

where $a_1, \ldots, a_n$ are positive constants. If the constants $a_1, \ldots, a_n$ are such that $a_1 + \cdots + a_n \leq 1$, then the product $Z_n = \xi_1 \cdots \xi_n$ is M-det.

Theorem 9. *Suppose that the growth rates $r_1, \ldots, r_n$ of the even order moments of the independent random variables $\xi_1, \ldots, \xi_n$ satisfy*

$$\frac{m_{1,2(k+1)}}{m_{1,2k}} = \mathcal{O}((k+1)^{r_1}), \ \ldots, \ \frac{m_{n,2(k+1)}}{m_{n,2k}} = \mathcal{O}((k+1)^{r_n}) \ \ as \ \ k \to \infty,$$

where $m_{i,2k} = \mathbf{E}[\xi_i^{2k}], i = 1, \ldots, n, \ k = 1, 2, \ldots$. If the rates $r_1, \ldots, r_n$ are such that $r_1 + \cdots + r_n \leq 2$, then the product $Z_n = \xi_1 \cdots \xi_n$ is M-det.

Let us describe now conditions under which the product Z_n is M-indet.

Theorem 10. *Consider n independent random variables $\xi_i \sim F_i$, $i = 1, \ldots, n$, where $n \geq 2$. Suppose each F_i has a density f_i which is strictly positive on R and symmetric about 0. Assume further that the following conditions are satisfied:*
(i) At least one of the densities $f_1(x), \ldots, f_n(x)$ is decreasing in $x \geq x_0$, where $x_0 \geq 1$ is a constant.
(ii) For each $i = 1, \ldots, n$, there exists a constant $A_i > 0$ such that (1) holds and there exist constants $B_i > 0$, $\alpha_i > 0$, $\beta_i > 0$ and real γ_i such that (2) holds.
If, in addition to the above, $\sum_{i=1}^n 1/\beta_i > 1$, then the product $Z_n = \xi_1 \cdots \xi_n$ is M-indet.

Example 3. We now apply Theorem 10 to the product of double generalized gamma random variables. We write $\xi \sim DGG(\alpha, \beta, \gamma)$ if ξ is a random variable in R with density function of the form

$$f(x) = c|x|^{\gamma - 1} \exp(-\alpha|x|^\beta), \ \ x \in R.$$

Here $\alpha, \beta, \gamma > 0$, $f(0) = 0$ if $\gamma \neq 1$, and $c = \beta\alpha^{\gamma/\beta}/(2\Gamma(\gamma/\beta))$ is a norming constant.

Corollary 3. *Suppose that $\xi_1, \ldots, \xi_n$ are n independent random variables, and let $\xi_i \sim DGG(\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2, \ldots, n$. Then the product $Z_n = \xi_1 \cdots \xi_n$ is M-det if and only if $\sum_{i=1}^n 1/\beta_i \leq 1$.*

5. The Mixed Case

For completeness of our study we need to consider products of both types of random variables, nonnegative ones and real ones. Since such a 'mixed' product takes values in R, this is a Hamburger case, so we can formulate results similar to Theorems 8 and 9. Since the conditions, the statements and the arguments are almost as in these two theorems, we do not give details. Instead, we present now a result in which the 'mixed' product $Z_n = \xi_1 \cdots \xi_n$ is M-indet.

Theorem 11. *Given are n independent random variables, such that the 'first' group, $\xi_1, \ldots, \xi_{n_0}$, consists of nonnegative variables, while the variables in the 'second' group, $\xi_{n_0+1}, \ldots, \xi_n$, all take values in R, where $1 \leq n_0 < n$. Suppose each $\xi_i \sim F_i$ has a density f_i and assume further that f_i, $i = 1, \ldots, n_0$, are strictly positive on $(0, \infty)$, while f_j, $j = n_0 + 1, \ldots, n$, are strictly positive on R and symmetric about 0. Moreover, assume the following conditions are satisfied:*
(i) At least one of the densities $f_j(x)$, $j = n_0 + 1, \ldots, n$, is decreasing in $x \geq x_0$, where $x_0 \geq 1$ is a constant.
(ii) For each $i = 1, \ldots, n$, there exists a constant $A_i > 0$ such that (1) holds and there exist constants $B_i > 0$, $\alpha_i > 0$, $\beta_i > 0$ and real γ_i such that (2) holds.
If, in addition to (i) and (ii), the parameters β_i are such that $\sum_{i=1}^n 1/\beta_i > 1$, then the product $Z_n = \xi_1 \cdots \xi_n$ is M-indet.

As an application of Theorem 11 we derive below two interesting corollaries.

Corollary 4. *Consider two independent random variables, ξ and η, where $\xi \sim Exp(1)$ and $\eta \sim \mathcal{N}(0, 1)$ (standard normal). Then $Z = \xi\eta$ is M-indet.*

Corollary 5. *(i) The product of two independent random variables, one chi-square and one normal, is M-indet.*
(ii) The product of two independent random variables, one inverse Gaussian and one normal, is M-indet.

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