Model of fatigue failure due to equicorrelated multiple cracks using extended Birnbaum-Saunders distribution

Ricardo Leiva
F.C.E. Universidad Nacional de Cuyo, Mendoza, Argentina - rleiva@fcemail.uncu.edu.ar

Anuradha Roy*
The University of Texas at San Antonio, San Antonio, USA - Anuradha.Roy@utsa.edu

Rubén Bageta
F.C.A. Universidad Nacional de Cuyo, Mendoza, Argentina - cbageta@fca.uncu.edu.ar

Juan Carlos Pina
Eindhoven University of Technology, Eindhoven, The Netherlands - J.C.Pina@tue.nl

Abstract

Statement: We propose an extension of the Birnbaum-Saunders distribution to model the more realistic situations where the fatigue failure time of a material is due to the growth of multiple cracks. The properties of the extended Birnbaum-Saunders (EBS) distribution are provided and its purpose to model the fatigue failure time of a material due to the growth of equicorrelated multiple cracks is proposed.
Discussion: The moment estimates (MEs) and the maximum likelihood estimates (MLEs) of the unknown parameters of the EBS distribution based on independent but not identically distributed (i.n.i.d.) random variables are obtained. Monte Carlo (MC) simulations are done to assess the performance of the parameter estimators of the EBS distribution with different number of cracks for different samples.
Summary: Comparisons of the parameter estimates and their standard deviations are made between this new EBS distribution with i.n.i.d. samples and the EBS distribution with i.i.d. samples.
Conclusions: Applications of EBS distribution for i.n.i.d. random variables are feasible and useful in real lile.

Keywords: maximum likelihood estimators; moment type estimators; Monte Carlo simulations.

1. Introduction

Fatigue life distribution model related to the reliability of a truck engine is derived and a solution is found by estimating the unknown parameters in the model. Leiva et al.(2015) proposed an EBS distribution to model the lifetime of a material subject to periodic loading, which cause the growth of several cracks (m) until the size of one of these cracks exceeds its critical threshold, which activates the failure of a system. They introduced an EBS distribution to model multiple cracks with $\boldsymbol{A}$ and $\boldsymbol{\beta}$ as the shape matrix and the scale vector respectively, as compared to the scalar quantities α and β as the shape and the scale parameters of the traditional BS distribution ($m = 1$) (Lemonte et al. (2007)). Leiva et al.(2015) assumed that the number of cracks is constant for all the samples, but in reality the number of cracks would vary from individual sample to sample. This article introduces an EBS distribution with possible different number of cracks (m_i) for different EBS random samples. EBS distribution was motivated by the thermo-mechanical fatigue (TMF) failure of solid materials with heterogeneous complex microstructures which are ubiquitous in nature. TMF arises due to thermal related stresses that take place during normal operating conditions. A typical example of TMF failure can be found in the cylinder head of truck engines which are subject to frequent temperature changes. The increase in temperature makes the material to expand. When the engine is shut down, the temperature decreases and compressive stress are relieved. The repetition of the "start up - shut down" cycle of the engine could lead to localized multiple cracking (m) of the truck engine cylinder head. Finally, the m microcracks form multiple larger cracks that ultimately lead to failure of the system.

2. Definition of the extended Birnbaum-Saunders distribution to model multiple cracks

The EBS distribution is defined as follows:

Definition 1. *Given the m-variate vector* $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$, *with* $\beta_j > 0$ *for* $j = 1, \ldots, m$, *and given the positive definite matrix* $\boldsymbol{A} = (a_{jh})_{j,h=1}^{m}$, *with* $a_{jh} \geq 0$ *for* $j, h = 1, \ldots, m$, *the non negative random variable* T *has the extended Birnbaum-Saunders (EBS) distribution with parameters* $\boldsymbol{A}$ *and* $\boldsymbol{\beta}$ *and it is denoted by* $T \sim EBS_m(\boldsymbol{A}, \boldsymbol{\beta})$, *if its distribution function* F_T *is given by*

$$F_T(t) = P[T \leq t] = 1 - \Phi_m(\boldsymbol{A} \cdot \boldsymbol{r}(t))\, I_{\Re_+}(t),$$

where $I_{\Re_+}$ *is the indicator function of the set of the positive real numbers* $\Re_+$, *that is,* $I_{\Re_+}(t) = \begin{cases} 1 & if \quad t \epsilon \Re_+ \\ 0 & if \quad t \notin \Re_+ \end{cases}$, *and where* $\boldsymbol{r} : \Re_+ \longrightarrow \Re^m$ *is the real vectorial function given by*

$$\boldsymbol{r}(t) = \boldsymbol{r}_{m,\boldsymbol{\beta}}(t) = (r_{1,\beta_1}(t), \ldots, r_{m,\beta_m}(t))' = \left(\sqrt{\tfrac{\beta_1}{t}} - \sqrt{\tfrac{t}{\beta_1}} \quad , \ldots, \quad \sqrt{\tfrac{\beta_m}{t}} - \sqrt{\tfrac{t}{\beta_m}} \right)'.$$

For detail see Leiva et al. (2015).

Property 1. *Let* $T \sim EBS_m(\boldsymbol{A}; \boldsymbol{\beta})$, *where* $\boldsymbol{A} = \left(\underset{1\times m}{\boldsymbol{a}_i} \right)_{i=1}^{m}$, *and* $\boldsymbol{\beta} = \mathbf{1}_n \otimes \beta_1$, *with a positive real number* β_1. *Then* $T^{-1} \overset{d}{=} \max\left\{T_1^{-1}, \ldots, T_m^{-1}\right\}$, *where* $T_1^{-1}, \ldots, T_m^{-1}$ *are independent* $EBS_1\left(b_i; \beta_1^{-1}\right)$, *with* $b_i = \boldsymbol{a}_i \mathbf{1}_m = \sum_{j=1}^{m} a_{i,j}$ *for* $i = 1, \ldots, m$.

Proof. Proof is straight forward.

3. Moment type estimates of ρ and β

From now on we consider the special case that $T \sim EBS_m(\boldsymbol{A}; \boldsymbol{\beta})$, where $\boldsymbol{A}$ is the equicorrelated shape matrix $\boldsymbol{A} = (1-\rho)\boldsymbol{I}_m + \rho \boldsymbol{J}_m$, with $0 < \rho < 1$ and where $\boldsymbol{\beta} = \beta \mathbf{1}_m$, the constant scale vector. We define $\theta = 1 + (m-1)\rho > 0$. By Part (b) of the Proposition 1 in Leiva et al. (2015) we know that the random variable T can be expressed as a function of the random variable

$$U = -\theta r_\beta(T), \tag{1}$$

with

$$T = r_\beta^{-1}\left(-\frac{U}{\theta}\right) = \frac{\beta}{4}\left[\frac{U}{\theta} + \sqrt{\frac{U^2}{\theta^2} + 4}\right]^2 = \frac{\beta}{2\theta^2}\left[U^2 + 2\theta^2 + U\sqrt{U^2 + 4\theta^2}\right],$$

where the c.d.f. of the random variable U is given in the next proposition.

Proposition 1. *The random variable* U *has the same distribution as the* $U_{(1,m)} = \min\{U_1, \ldots, U_m\}$, *where* $U_1, \ldots, U_m \overset{i.i.d.}{\sim} N_1(0,1)$, *that is,* $F_U(u)$ *is given by* $F_U(u) = 1 - [1 - \Phi_1(u)]^m$.

Proof. Since $r_\beta(t)$ is a strictly decreasing function and $\theta > 0$, we have

$$\begin{aligned} F_U(u) &= P[U \leq u] = P\left[r_\beta(T) \geq -\frac{u}{\theta}\right] = P\left[T \leq r_\beta^{-1}\left(-\frac{u}{\theta}\right)\right] \\ &= 1 - \left[1 - \Phi_1\left(-\theta r_\beta\left(r_\beta^{-1}\left(-\frac{u}{\theta}\right)\right)\right)\right]^m = 1 - \left[1 - \Phi_1(u)\right]^m. \end{aligned}$$

Similarly, using Property 1 of Section 2 we show that for the equicorrelated case of $T \sim EBS_m(\boldsymbol{A}; \boldsymbol{\beta})$, the distribution function of T^{-1} is $F_{T^{-1}}(x) = \left[\Phi_1\left(\theta r_{\beta^{-1}}\left(x\right)\right)\right]^m$, and the following result holds.

Proposition 2. *The random variable*

$$U^* = -\theta r_{\beta^{-1}}\left(T^{-1}\right) \tag{2}$$

has the same distribution as that of the $U_{(m,m)} = \max\{U_1, \ldots, U_m\}$, *where* $U_1, \ldots, U_m \overset{i.i.d.}{\sim} N_1(0,1)$, *that is,* $F_{U^*}(u)$ *is given by* $F_{U^*}(u) = [\Phi_1(u)]^m$.

Proof. Since $r(t)$ is a strictly decreasing function and $\theta > 0$, we have

$$\begin{aligned} F_{U^*}(u) &= P[U^* \le u] = P\left[r_{\beta^{-1}}\left(T^{-1}\right) \ge -\frac{u}{\theta}\right] = P\left[T^{-1} \le r_{\beta^{-1}}^{-1}\left(-\frac{u}{\theta}\right)\right] \\ &= \left[\Phi_1\left(-\theta r_{\beta^{-1}}\left(r_{\beta^{-1}}^{-1}\left(-\frac{u}{\theta}\right)\right)\right)\right]^m = \left[\Phi_1(u)\right]^m, \end{aligned}$$

that is, $U^* \stackrel{d}{=} U_{(m,m)} = \max\{U_1, \ldots, U_m\}$, where $U_1, \ldots, U_m \stackrel{i.i.d.}{\sim} N_1(0,1)$.
Another way of proving the above result is first noting that $V = -r_{\beta^{-1}}\left(T^{-1}\right) = r_\beta(T)$, and then

$$\begin{aligned} U^* &= -\theta r_{\beta^{-1}}\left(T^{-1}\right) = \theta r_\beta(T) = -U \\ &= -\min\{U_1, \ldots, U_m\} = \max\{U_1^*, \ldots, U_m^*\} = U_{(m,m)}, \end{aligned}$$

where $U_1^*, \ldots, U_m^* \stackrel{i.i.d.}{\sim} N_1(0,1)$. Now, Proposition 1 allows to relate properties of $T \sim EBS_m(\boldsymbol{A}, \boldsymbol{\beta})$, where $\boldsymbol{A} = (1-\rho)\boldsymbol{I}_m + \rho \boldsymbol{J}_m$ and $\boldsymbol{\beta} = \beta \mathbf{1}_m$, with properties of $U_{(1,m)} = U \stackrel{d}{=} \min\{U_1, \ldots, U_m\}$, where $U_1, \ldots, U_m \stackrel{i.i.d.}{\sim} N_1(0,1)$, through the equality

$$T = \frac{\beta}{2\theta^2}\left[U^2 + 2\theta^2 + U\sqrt{U^2 + 4\theta^2}\right].$$

In particular, we can find expressions (as a function of θ and β) of the population mean $E[T]$ based on the population raw moments (moments about 0) of $U \stackrel{d}{=} U_{(1,m)} = \min\{U_1, \ldots, U_m\}$. Taking expectation of both sides of the above equation, we have

$$E[T] = \frac{\beta}{2\theta^2} E\left[U_{(1,m)}^2 + 2\theta^2 + \sqrt{U_{(1,m)}^4 + 4\theta^2 U_{(1,m)}^2}\right] = \frac{\beta}{2\theta^2} E\left[h_\theta\left(U_{(1,m)}^2\right)\right], \tag{3}$$

$$\text{where} \quad h_\theta(Z) = Z^2 + 2\theta^2 + \sqrt{\left(Z^2\right)^2 + 4\theta^2 Z^2}. \tag{4}$$

Similarly, Proposition 2 allows to relate the properties of the equicorrelated case of $T \sim EBS_m(\boldsymbol{A}, \boldsymbol{\beta})$ with properties of $U^* \stackrel{d}{=} U_{(m,m)} = \max\{U_1, \ldots, U_m\}$, where $U_1, \ldots, U_m \stackrel{i.i.d.}{\sim} N_1(0,1)$ through the equality

$$T^{-1} = r_{\beta^{-1}}^{-1}\left(-\frac{U^*}{\theta}\right) = \frac{\beta^{-1}}{2\theta^2}\left[U^{*2} + 2\theta^2 + U^*\sqrt{U^{*2} + 4\theta^2}\right], \quad \text{and then}$$

$$E\left[T^{-1}\right] = \frac{\beta^{-1}}{2\theta^2} E\left[U_{(m,m)}^2 + 2\theta^2 + \sqrt{U_{(m,m)}^4 + 4\theta^2 U_{(m,m)}^2}\right] = \frac{\beta^{-1}}{2\theta^2} E\left[h_\theta\left(U_{(m,m)}^2\right)\right], \tag{5}$$

where $h_\theta(Z)$ is given in (4). Since for the standard Normal case $U_{(1,m)}^2 \stackrel{d}{=} U_{(m,m)}^2$, we have

$$E\left[h_\theta\left(U_{(1,m)}^2\right)\right] = E\left[h_\theta\left(U_{(m,m)}^2\right)\right]. \tag{6}$$

This equality (6), and the equalities (3) and (5) allow us to find in the equicorrelated case a moment type estimator $\widetilde{\beta}$ of β based on a random sample $T_1, \ldots, T_n \stackrel{i.i.d.}{\sim} EBS_m(\boldsymbol{A}, \boldsymbol{\beta})$, because then we have $\frac{2\theta^2}{\beta} E[T] = \frac{2\theta^2}{\beta^{-1}} E\left[T^{-1}\right]$, i.e., $\beta^2 = \frac{E[T]}{E[T^{-1}]}$. Replacing $E[T]$ and $E\left[T^{-1}\right]$ respectively by $M'_{1,T} = \frac{1}{n}\sum_{j=1}^n T_j = \overline{T}$, and $M'_{1,T^{-1}} = \frac{1}{n}\sum_{j=1}^n T_j^{-1} = H^{-1}$ where H is the harmonic sample mean, we obtain

$$\widetilde{\beta} = \left(\frac{M'_{1,T}}{M'_{1,T^{-1}}}\right)^{\frac{1}{2}} = \left(\overline{T} \cdot H\right)^{\frac{1}{2}}. \tag{7}$$

This is the same modified moment estimate proposed by Ng. et al. (2003) for $m = 1$. To find a moment type estimator $\widetilde{\theta}$ of θ we need another (independent) equation. From (2) and Proposition 2 we know that $U_{(m,m)} = -\theta r_{\beta^{-1}}\left(T^{-1}\right)$. Therefore

$$\frac{U_{(m,m)}}{\theta} \stackrel{d}{=} -r_{\beta^{-1}}\left(T^{-1}\right) = r_\beta(T) = \beta^{\frac{1}{2}} T^{-\frac{1}{2}} - \beta^{-\frac{1}{2}} T^{\frac{1}{2}},$$

and $\frac{U^2_{(m,m)}}{\theta^2} + 2 \stackrel{d}{=} \beta^{-1}T + \beta T^{-1}$. Then taking expectation of both sides of the above equation and replacing the first population moments of T and T^{-1} by their corresponding sample moments, we obtain

$$E\left[U^2_{(m,m)}\right]\frac{1}{\theta^2} = M'_{1,T}\beta^{-1} + M'_{1,T^{-1}}\beta - 2.$$

(Note that the same equation is obtained starting from (1) and Proposition 1 due to the equality $E\left[U^2_{(1,m)}\right] = E\left[U^2_{(m,m)}\right]$). Now, substituting the value of $\widetilde{\beta}$ from (7), the solution $\widetilde{\theta}$ of the above equation is

$$\widetilde{\theta} \quad = \quad \left\{\frac{E\left[U^2_{(m,m)}\right]}{M'_{1,T}\widetilde{\beta}^{-1} + M'_{1,T^{-1}}\widetilde{\beta} - 2}\right\}^{\frac{1}{2}} = \left\{\frac{E\left[U^2_{(m,m)}\right]}{2\left[\left(M'_{1,T}M'_{1,T^{-1}}\right)^{\frac{1}{2}} - 1\right]}\right\}^{\frac{1}{2}}. \tag{8}$$

Finally, from this expression (8), for $m > 1$, the moment type estimate of ρ is given by

$$\widetilde{\rho} = \begin{cases} 0 & if \quad \widetilde{\theta} < 1 \\ \frac{\widetilde{\theta}-1}{m-1} & if \quad 1 \leq \widetilde{\theta} \leq m \\ 1 & if \quad \widetilde{\theta} > m \end{cases}. \tag{9}$$

For the effective calculation of $\widetilde{\rho}$ (as well as that of $\widetilde{\theta}$), values of $E\left[U^2_{(m,m)}\right]$ are needed; we obtain them by applying the 128 point Gauss-Hermite quadrature formula using Matlab Release 6. In any real life applications the usual assumptions is that the sample variables are i.i.d. However, when all the variables do not have the same number of cracks, one cannot assume the samples are identically distributed while using the EBS distribution for material fatigue studies. Therefore, $T_1, \ldots, T_n$ are i.n.i.d. with $T_i \sim EBS_{m_i}\left(\boldsymbol{A}_i, \boldsymbol{\beta}_i\right) = EBS_{m_i}\left(\left(1-\rho\right)\boldsymbol{I}_{m_i} + \rho\boldsymbol{J}_{m_i}, \beta\mathbf{1}_{m_i}\right)$ for $i = 1, \ldots, n$. However, note that the structures of the shape matrix $\boldsymbol{A}$ and the scale vector $\boldsymbol{\beta}$, and consequently the parameters ρ and β remain the same. To find the moment type estimates under the i.n.i.d. random variables, let $t_1, \ldots, t_n$ be a sample of size n such that t_i is a realization of $T_i \sim EBS_{m_i}\left(\boldsymbol{A}_i, \boldsymbol{\beta}_i\right)$ with $\boldsymbol{A}_i = (1-\rho)\boldsymbol{I}_{m_i} + \rho\boldsymbol{J}_{m_i}$, $\boldsymbol{\beta}_i = \beta\mathbf{1}_{m_i}$, and thus $\theta_i = 1 + (m_i - 1)\rho > 0$ for each $i = 1, \ldots, n$, with fixed known natural numbers $m_1, \ldots, m_n \in \{2, \ldots, m\}$, where $m = \max\{m_1, \ldots, m_n\}$. We propose a moment type estimate $\widetilde{\beta}$ of β based on the values $t_1, \ldots, t_n$ from i.n.i.d. random variables as

$$\widetilde{\beta} = \frac{1}{n}\sum_{i=1}^{n}\widetilde{\beta}_i = \frac{1}{n}\sum_{i=1}^{n}t_i = \bar{t}. \tag{10}$$

On the other hand, moment type estimator $\widetilde{\theta}$ of θ in (8) cannot be used directly when we have only one realization t_i of T_i because the denominator becomes zero. However, we can use (10) to first replace $\widetilde{\beta}$ in (8) and then replace $M'_{1,T}$ and $M'_{1,T^{-1}}$ by t_i and t_i^{-1}, respectively to obtain

$$\widetilde{\theta}_i \quad = \quad \left\{\frac{E\left[U^2_{(m_i,m_i)}\right]}{t_i\widetilde{\beta}^{-1} + t_i^{-1}\widetilde{\beta} - 2}\right\}^{\frac{1}{2}} = \left\{\frac{E\left[U^2_{(m_i,m_i)}\right]}{\left[r_{\widetilde{\beta}}\left(t_i\right)\right]^2}\right\}^{\frac{1}{2}} = \frac{E\left[U^2_{(m_i,m_i)}\right]^{\frac{1}{2}}}{\left|r_{\widetilde{\beta}}\left(t_i\right)\right|}, \quad \text{and}$$

$$\widetilde{\rho}_i \quad = \quad \begin{cases} 0 & if \quad \widetilde{\theta}_i < 1 \\ \frac{\widetilde{\theta}_i-1}{m_i-1} & if \quad 1 \leq \widetilde{\theta}_i \leq m_i \\ 1 & if \quad \widetilde{\theta}_i > m_i \end{cases},$$

for each $i = 1, \ldots, n$. We now propose a moment type estimate $\widetilde{\rho}$ of ρ based on the realizations $t_1, \ldots, t_n$ from i.n.i.d. random variables $T_i \sim EBS_{m_i}\left(\boldsymbol{A}_i, \boldsymbol{\beta}_i\right) = EBS_{m_i}\left(\left(1-\rho\right)\boldsymbol{I}_{m_i} + \rho\boldsymbol{J}_{m_i}, \beta\mathbf{1}_{m_i}\right)$ for $i = 1, \ldots, n$ as

$$\widetilde{\rho} = \frac{1}{n}\sum_{i=1}^{n}\widetilde{\rho}_i. \tag{11}$$

In the next section we present the theory of the MLEs for the i.n.i.d. set up.

4. Maximum Likelihhod estimates of ρ and β

The MLEs $\widehat{\rho}$ and $\widehat{\beta}$ of the parameters ρ and β are obtained from n random samples $T_1, \ldots, T_n$, such that $T_i \sim EBS_{m_i}(\boldsymbol{A}_i, \boldsymbol{\beta}_i) = EBS_{m_i}((1-\rho)\boldsymbol{I}_{m_i} + \rho \boldsymbol{J}_{m_i}, \beta \boldsymbol{1}_{m_i})$, by finding the values $\widehat{\rho}$ and $\widehat{\beta}$ where the likelihood function $L = L(\rho, \beta)$ is maximized. The log-likelihood function is given by

$$\begin{aligned} \log L &= \log L(\rho,\beta) = \log L(\sigma_0, \sigma_1, \beta) \\ &= \sum_{i=1}^{n} \log \frac{m_i \left[1 + (m_i - 1)\rho\right]}{2} + \sum_{i=1}^{n} \log \varphi_1 \left(\left[1 + (m_i-1)\rho\right] r(t_i,\beta)\right) \\ &\quad + \sum_{i=1}^{n} (m_i - 1) \log \Phi_1 \left(\left[1 + (m_i-1)\rho\right] r(t_i,\beta)\right) - n \log \beta + \sum_{i=1}^{n} \log \left[\left(\frac{\beta}{t_i}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t_i}\right)^{\frac{3}{2}}\right], \end{aligned}$$

where $\theta_i = 1 + (m_i - 1)\rho$. Substituting the expressions of $\frac{dr(t,\beta)}{d\beta}, r(t,\beta)\frac{dr(t,\beta)}{d\beta}$ and $\frac{d}{d\beta}\left(\frac{\beta^{\frac{1}{2}}}{t^{\frac{1}{2}}} + \frac{\beta^{\frac{3}{2}}}{t^{\frac{3}{2}}}\right)$ from Leiva et al. (2015) in the derivative values in $\frac{\partial \log L}{\partial \rho}$ and $\frac{\partial \log L}{\partial \beta}$, and simplifying and equating these partial derivatives separately to 0, we get

$$0 = \sum_{i=1}^{n} (m_i - 1) \left\{ \frac{1}{\theta_i} - \theta_i \left[\frac{\beta}{t_i} + \frac{t_i}{\beta} - 2\right] + (m_i - 1) \frac{r(t_i,\beta)\varphi_1(\theta_i r(t_i,\beta))}{\Phi_1(\theta_i r(t_i,\beta))} \right\}, \quad \text{and} \tag{12}$$

$$0 = -\beta \sum_{i=1}^{n} \theta_i^2 \frac{1}{t_i} + \frac{1}{\beta} \sum_{i=1}^{n} \theta_i^2 t_i + \sum_{i=1}^{n} (m_i - 1)\theta_i \frac{\varphi_1(\theta_i r(t_i,\beta))}{\Phi_1(\theta_i r(t_i,\beta))} \left(\frac{\beta^{\frac{1}{2}}}{t_i^{\frac{1}{2}}} + \frac{t_i^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}\right) - n + 2\beta \sum_{i=1}^{n} \frac{1}{t_i + \beta}. \tag{13}$$

The MLEs $\widehat{\rho}$ and $\widehat{\beta}$ are obtained by simultaneously and iteratively solving the two equations (12) and (13). MC simulation is performed to find the MLEs $\widehat{\rho}$ and $\widehat{\beta}$ by using the 'Fmincon' subroutine of the MATLAB, which finds a constrained $\big($in our case $0 < \rho < 1$ and $\beta > 0\big)$ minimum of an objective function of several variables $\big(-\log\big(V(\rho,\beta)\big)\big)$ starting at an initial estimate (ρ_0, β_0) of (ρ, β). To facilitate the optimization, initial values β_0 and ρ_0 are taken as the moment type estimates $\widetilde{\beta}$ and $\widetilde{\rho}$ as given in (10) and (11) respectively. If $m_i = m$, and thus $\theta_i = \theta$, (13) reduces to the identical equation obtained in Leiva et al. (2015).

4. Simulation studies

In order to evaluate the performance of the ML estimators derived in Section 3, we perform MC simulations for different values ρ as 0.3, 0.5 and 0.9, for different parameter values $m_1, \ldots, m_n$, and for different sample sizes n, where $n = 10, 20, 50$ and 100. Since β is the scale parameter, without loss of generality β is kept fixed at $\beta = 1$. The values of bias and standard deviation of $\widehat{\beta}$ are simply needed to be multiplied by β (if it

Table 1: Means of $\widehat{\rho}$ and $\widehat{\beta}$ based on Monte Carlo simulation ($\beta = 1.0$)

ρ	n	$m=5$		$m=10$		$m=5$		$m=10$	
		$\widehat{\rho}$	$\widehat{\rho}_{\text{Const}}$	$\widehat{\rho}$	$\widehat{\rho}_{\text{Const}}$	$\widehat{\beta}$	$\widehat{\beta}_{\text{Const}}$	$\widehat{\beta}$	$\widehat{\beta}_{\text{Const}}$
0.3	10	0.3971	0.3905	0.3767	0.3663	0.9760	0.9706	0.9690	0.9730
	20	0.3425	0.3394	0.3341	0.3289	0.9847	0.9852	0.9864	0.9876
	50	0.3159	0.3146	0.3134	0.3100	0.9950	0.9929	0.9943	0.9946
	100	0.3079	0.3070	0.3055	0.3045	0.9972	0.9967	0.9970	0.9980
0.5	10	0.6187	0.5997	0.5950	0.5873	0.9773	0.9815	0.9811	0.9825
	20	0.5565	0.5504	0.5485	0.5416	0.9885	0.9880	0.9892	0.9911
	50	0.5238	0.5203	0.5138	0.5143	0.9947	0.9940	0.9970	0.9970
	100	0.5112	0.5095	0.5091	0.5059	0.9967	0.9972	0.9977	0.9988
0.9	10	0.9064	0.9050	0.9146	0.9092	1.0018	1.0032	0.9991	1.0010
	20	0.9034	0.9043	0.9045	0.9095	1.0025	1.0009	1.0008	0.9998
	50	0.9048	0.9043	0.9064	0.9082	0.9997	1.0001	0.9998	0.9996
	100	0.9059	0.9068	0.9055	0.9053	0.9996	0.9990	0.9994	0.9997

is different from 1), nevertheless the bias and the standard deviation of $\widehat{\rho}$ will not be affected. Table 1 compares the means of $\widehat{\rho}$ and $\widehat{\beta}$ for constant m, i.e., for i.i.d. samples, i.e., for $m_i = m$ as well as the means for i.n.i.d. samples for $i = 1, \ldots, n$ for $m = 5$ and $m = 10$ based on MC simulation ($\beta = 1.0$). We use the notations $\widehat{\rho}_{\text{Const}}$ and $\widehat{\beta}_{\text{Const}}$ to denote the values of $\widehat{\rho}$ and $\widehat{\beta}$ for constant m (Leiva et al. (2015)). The notations $\widehat{\rho}$ and $\widehat{\beta}$ are used for maximum m. We see the behaviors of $\widehat{\rho}$ and $\widehat{\beta}$ are very similar to the behaviors of $\widehat{\rho}_{\text{Const}}$ and $\widehat{\beta}_{\text{Const}}$, however $\widehat{\rho}_{\text{Const}}$ is a better estimate of the true value ρ, for small n and $\rho \leq 0.5$. For large n, both the estimates $\widehat{\rho}$ and $\widehat{\rho}_{\text{Const}}$ are mostly similar. This is expected as more samples means more information, thus better estimates. For $\rho > 0.5$, the estimates $\widehat{\rho}$ and $\widehat{\rho}_{\text{Const}}$ are mostly similar. This is expected as ρ increases the cracks become less and less diverse, even if the number of cracks are different for different samples (i.n.i.d. case). Increase in m provides better estimates for both $\widehat{\rho}$ and $\widehat{\rho}_{\text{Const}}$, as increase in m means more cracks, thus more information and ultimately better estimates. Table 2 compares the

Table 2: Standard deviations of $\widehat{\rho}$ and $\widehat{\beta}$ based on Monte Carlo simulation ($\beta = 1.0$)

ρ	n	$m = 5$		$m = 10$		$m = 5$		$m = 10$	
		$\widehat{\rho}$	$\widehat{\rho}_{\text{Const}}$	$\widehat{\rho}$	$\widehat{\rho}_{\text{Const}}$	$\widehat{\beta}$	$\widehat{\beta}_{\text{Const}}$	$\widehat{\beta}$	$\widehat{\beta}_{\text{Const}}$
0.3	10	0.1956	0.1757	0.1521	0.1342	0.1501	0.1371	0.1101	0.0931
	20	0.1131	0.1081	0.0871	0.0807	0.1019	0.0982	0.0749	0.0675
	50	0.0672	0.0631	0.0513	0.0458	0.0670	0.0632	0.0492	0.0424
	100	0.0479	0.0427	0.0346	0.0316	0.0482	0.0447	0.0350	0.0306
0.5	10	0.2169	0.2004	0.1812	0.1732	0.1127	0.0996	0.0706	0.0624
	20	0.1526	0.1410	0.1245	0.1149	0.0779	0.0704	0.0500	0.0445
	50	0.0899	0.0842	0.0703	0.0657	0.0495	0.0463	0.0330	0.0284
	100	0.0619	0.0576	0.0503	0.0453	0.0353	0.0318	0.0243	0.0200
0.9	10	0.1327	0.1321	0.1215	0.1219	0.0594	0.0541	0.0349	0.0289
	20	0.1170	0.1136	0.1065	0.1032	0.0450	0.0394	0.0250	0.0216
	50	0.0927	0.0900	0.0851	0.0815	0.0294	0.0271	0.0178	0.0154
	100	0.0731	0.0712	0.0677	0.0651	0.0216	0.0194	0.0137	0.0114

standard deviations (s.d.s) of $\widehat{\rho}$ and $\widehat{\beta}$ for constant m, i.e., for i.i.d. samples, i.e., for $m_i = m$ and the s.d.s for i.n.i.d. samples for all $i = 1, \ldots, n$ for $m = 5$ and $m = 10$ based on MC simulation ($\beta = 1.0$). We see the behaviors of s.d.s of $\widehat{\rho}$ and $\widehat{\beta}$ are very similar to the behaviors of s.d.s of $\widehat{\rho}_{\text{Const}}$ and $\widehat{\beta}_{\text{Const}}$, however s.d.s of $\widehat{\rho}_{\text{Const}}$ is less than the s.ds of $\widehat{\rho}$ for small n and for $\rho \leq 0.5$. For large n, the s.d.s of $\widehat{\rho}$ and $\widehat{\rho}_{\text{Const}}$ are mostly similar; also the s.d.s of $\widehat{\beta}$ and $\widehat{\beta}_{\text{Const}}$ are mostly similar. This is expected as more samples means more information and less s.d.s. For $\rho > 0.5$, the s.d.s of $\widehat{\rho}$ and $\widehat{\rho}_{\text{Const}}$ are mostly similar. This is expected as ρ increases the cracks become less and less diverse. Increase in m gives less s.d.s for both $\widehat{\rho}$ and $\widehat{\rho}_{\text{Const}}$ too. This is also expected, as increase in m means more cracks, thus more information and less s.d.s.

5. Conclusions

The authors address the issue of modeling in the analysis of common fatigue life problems motivated by real-world applications such as complex engine systems. This new EBS distribution has many applications on many realistic situations of fatigue life distributions today and many more in the coming days as well.

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