AN UPDATED BAYESIAN RELIABILITY ANALYSIS BASED ON STEP-STRESS ACCELERATED DEGRADATION DATA VIA GAMMA PROCESSES

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Abstract

Degradation analysis is more efficient than the conventional life test in drawing reliability assessment for high quality products. This research aims on the Bayesian approach to the step-stress accelerated degradation test when the degradation data of different products are collected via independent gamma processes. Based on the accelerated data, the prior distributions are updated when more date are observed by times for a new similar product under normal use condition. Sequential reliability inference will be made based on the updated posterior distribution of the underlying parameters with the aid of Markov chain Monte Carlo method. Simulation study and an illustrative example will be presented to show the appropriateness of the proposed method.
Keywords: Step-stress accelerated degradation test; Gamma process, reliability inference, Bayesian approach, MCMC

1. Introduction

For highly reliable products, it is quite difficult to obtain the products lifetimes through traditional life tests within a reasonable period of time. Even though one may use accelerated life tests by testing the products at higher levels of stress than the typical use condition (such as elevated temperatures or voltages), these methods provide little assistance because at most a few failures are likely to occur within a reasonable testing duration. Alternatively, accelerated degradation tests (ADTs) are widely used to assess the lifetime information of highly reliable products possessing quality characteristics that both degrade over time and can be related to reliability. In such a situation, one must identify a quality characteristic whose degradation over time can be related to reliability, then the products lifetime can be estimated well by collecting such degradation data. Detailed discussions on degradation models and their applications can be found in Nelson (1990), Meeker and Escobar (1998), and Wang (2008).

The performance of a degradation test strongly depends on the suitability of the assumed model of a products degradation path. For degradation paths involving -independent nonnegative increments, gamma processes are more suitable for describing the deterioration of the product. Bagdonavicius and Nikulin (2001) modeled traumatic events by a gamma process, and discussed possibly time-dependent covariates. Lawless and Crowder (2004) constructed a tractable gamma process by incorporating random effects. For some recent applications of gamma degradation models, see Noortwijk (2009) and Tseng et al. (2009).

In this article, we propose a Bayesian approach to the step-stress accelerated degradation test when the degradation data of products are collected via independent gamma processes. Reliability inference of the population will be made based on the posterior distribution of the underlying parameters with the aid of Markov chain Monte Carlo method. Our primary goal is to make predictive inference of a new but similar product sequentially as the new data is collected under normal use condition one by one. In addition to making reliability inference, we have also discussed how to terminate the experiment by specifying some standard based on the accuracy in estimating the mean time to failure as well as the quantiles of the lifetime distribution under normal use condition.

The rest of this paper is organized as follows. Section 2 presents the statistical formulation of the model. Bayesian inference of the lifetime distribution and the Bayesian sequential plan are discussed in Section 3. Section 4 provides the data analysis of a simulated data set. Finally, some concluding remarks are made in Section 5.

2. Section 2

Assume the degradation path $Y(t)$ follows a gamma process which satisfies

1. $Y(0) = 0$.
2. $Y(t)$ is of the independent increments, i.e. $Y(t) - Y(s)$ is independent of $Y(s)$, for $t > s > 0$.
3. $Y(t) - Y(s)$ is Gamma$(\alpha(t-s), \lambda)$ random variable, for $t > s > 0$ and $\alpha, \lambda > 0$.

Consider that n testing products are put under a step-stress degradation test. Assume that temperature($°C$) is the accelerating variable with levels S_j, $j = 1, \ldots, m$ and it follows the Arrhenius model. Specifically, let $Y_i(t)$ be the gamma degradation paths for products $i = 1, \ldots, n$ and $Y_{ijk} = Y_{ij}(t_k)$ be observed at $0 = t_0 < t_1 < \ldots, t_K$ under stress levels S_j over $0 \leq t_{\zeta_{j-1}} \leq t < t_{\zeta j} \leq t_K$, respectively for $j = 1, 2, \ldots, m$, and $i = 1, \ldots, n$. Define $\Delta Y_{ijk} = Y_{ij}(t_k) - Y_{ij}(t_{k-1})$ and $\Delta t_k = t_k - t_{k-1}$ (with $Y_{i1}(0) = t_0 = 0$). Thus, we have, for $k = \zeta_{j-1}+1, \ldots, \zeta_j$, $j = 1, \ldots, m$, $i = 1, \ldots, n$, $\Delta Y_{ijk} \sim$ Gamma$(\alpha_j \Delta t_k, \lambda)$ where $\alpha_j = \exp\left\{a + \frac{b}{273+S_j}\right\}$. Then given $\mathbf{S} = (\mathbf{S_1}, \mathbf{S_2}, \cdots, \mathbf{S_m})$, $\mathbf{\Delta t} = (\mathbf{\Delta t_1}, \mathbf{\Delta t_2}, \cdots, \mathbf{\Delta t_{\zeta_m}})$, and the observed degradation data $\mathbf{\Delta y} = \{\mathbf{\Delta y_{ijk}}\}$, for $k = \zeta_{j-1} + 1, \cdots, \zeta_j$, $j = 1, 2, \cdots m$, $i = 1, 2, \cdots, n$, the likelihood function of the unknown parameters $\theta = (\mathbf{a}, \mathbf{b}, \lambda)$ is

$$L(\theta|\mathbf{\Delta y}, \mathbf{S}, \mathbf{\Delta t}) \propto \prod_{\mathbf{i=1}}^{\mathbf{n}} \prod_{\mathbf{j=1}}^{\mathbf{m}} \prod_{\mathbf{k=\zeta_{j-1}+1}}^{\zeta_\mathbf{j}} \frac{\lambda^{\alpha_\mathbf{j} \mathbf{\Delta t_k}}}{\mathbf{\Gamma}(\alpha_\mathbf{j} \mathbf{\Delta t_k})} (\mathbf{\Delta y_{ijk}})^{\alpha_\mathbf{j} \mathbf{\Delta t_k - 1}} \mathbf{e}^{-\lambda \mathbf{\Delta y_{ijk}}}.$$

The major concern in reliability analysis is about the lifetime distribution under normal use condition S_0. Note that under S_0, the product fails if and only if the degradation path touches a given threshold Y_f. Thus, the cdf of the failure time T under S_0 is

$$F(t|\theta, \mathbf{S_0}) = \mathbf{P}(\mathbf{T} \leq \mathbf{t}|\theta, \mathbf{S_0}) = \mathbf{P}(\mathbf{Y(t)} > \mathbf{Y_f}|\theta, \mathbf{S_0}) = \frac{\mathbf{\Gamma}(\alpha_\mathbf{0}\mathbf{t}, \lambda \mathbf{Y_f})}{\mathbf{\Gamma}(\alpha_\mathbf{0}\mathbf{t})}, \tag{1}$$

where $t > 0$, $\alpha_0 = \exp\{a + b/(273 + S_0)\}$, and $\Gamma(\alpha, \beta) = \int_\beta^\infty \xi^{\alpha-1} e^{-\xi} d\xi$, the incomplete gamma function. To avoid computation of the incomplete gamma function, (1) can be approximated by the Birnbaum-Saunders distribution (BS distribution) introduced by Park and Padgett (2005) such as

$$F(t|\theta, \mathbf{S_0}) \approx \mathbf{\Phi}\left(\frac{\mathbf{1}}{\mathbf{A}}\left[\sqrt{\frac{\mathbf{t}}{\mathbf{B}}} - \sqrt{\frac{\mathbf{B}}{\mathbf{t}}}\right]\right), \quad \mathbf{t} > \mathbf{0},$$

where $A = 1/\sqrt{\lambda Y_f}$, $B = \lambda Y_f/\alpha_0$, and $\Phi(\cdot)$ is cdf of standard normal distribution.
Further inference of particular interest under S_0 of T includes the mean time to failure

$$MTTF_0(\theta) = \mathbf{E}(\mathbf{T}|\theta, \mathbf{S_0}) = \int_\mathbf{0}^\infty (\mathbf{1} - \mathbf{F}(\mathbf{t}|\theta, \mathbf{S_0}))\mathbf{dt} \approx \frac{\lambda \mathbf{Y_f}}{\alpha_\mathbf{0}} + \frac{\mathbf{1}}{\mathbf{2}\alpha_\mathbf{0}},$$

the reliability function

$$R(t|\theta, \mathbf{S_0}) = \mathbf{1} - \frac{\mathbf{\Gamma}(\alpha_\mathbf{0}\mathbf{t}, \lambda \mathbf{Y_f})}{\mathbf{\Gamma}(\alpha_\mathbf{0}\mathbf{t})} \approx \mathbf{1} - \mathbf{\Phi}(\frac{\mathbf{1}}{\mathbf{A}}[\sqrt{\frac{\mathbf{t}}{\mathbf{B}}} - \sqrt{\frac{\mathbf{B}}{\mathbf{t}}}]), \quad \mathbf{t} > \mathbf{0}.$$

and the p-Quantile, $0 < p < 1$,

$$t_p(\theta, \mathbf{S_0}) = \mathbf{F^{-1}}(\mathbf{p}|\theta, \mathbf{S_0}) \approx \begin{cases} (G-H)/2, & 0 \leq p < 0.5, \\ G/2, & p = 0.5, \\ (G+H)/2, & 0.5 < p \leq 1, \end{cases}$$

where $G = B[2 + (A\ \Phi^{-1}(p))^2]$, $H = AB\ \Phi^{-1}(p)\sqrt{4 + (A\ \Phi^{-1}(p))^2}$, with $\approx$ approximated by the BS distribution.

3. Section 3

We consider the Bayesian approach by using independent priors for the parameters a, b, and λ. Specifically, consider $N(\mu_a, \sigma_a^2)$ and $N(\mu_b, \sigma_b^2)$ priors for a and b, respectively, as they are the regression coefficients

associated with the stress variable, and a *Gamma*(c, d) is used for the nuisance scale parameter λ. A Markov chain Monte Carlo method is carried out to draw an approximate posterior sample of θ based on the Metropolis-Hastings algorithm and the Gibbs sampler. The posterior inference on the parameters of interest and/or functions of parameters can be drawn based on the approximate sample obtained from the MCMC procedure. For example, the posterior mean and posterior variance can be approximated by the corresponding sample mean and sample variance. Moreover, an approximate $100(1-\alpha)\%$ credible interval can also be obtained based on the corresponding lower and upper sample $\alpha/2$-quantiles. Furthermore, substituting the MCMC sample of the unknown parameters into $MTTF_0$, $R(t|S_0)$ as well as $t_p(S_0)$ which in turn can be used to develop the corresponding Bayesian inference.
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In reliability analysis, it is often more interested in making predictive inference for a new product. Suppose a new product but similar to the existing products is under investigation and a degradation test is made under normal use condition in order to draw reliability inference.
Let $z(t)$ be the degradation path of the new product under S_0 which is a gamma process with $\delta z = z(t + \triangle t) - z(t)$ follows the Gamma$(\alpha_z \triangle t, \lambda_z)$ distribution, where $\alpha_z = \exp\{a_z + b_z/(273 + S_0)\}$. Denoting $\theta_z = (a_z, b_z, \lambda_z)$ as the underlying parameter of the degradation model in $z(t)$ and assume that θ_z has the same form of prior distribution as θ. If degradation data are collected at $0 < t_1 < t_2 < \cdots$, one major concern is how long the experiment must be conducted to get reliable lifetime prediction. Base on previous results, we may receive the information from the previous SSADT data as the prior input for the beginning of the experiment. On the other hand, as degradation data are taken sequentially observed as time goes by, more precise information can be updated from time to time. In this article, we update the information through the parameters of the prior distributions so that the posterior of θ_z at time t_k can be updated by incorporating with the new observation $z(t_k)$ as well as the updated prior based on all previous observations $\mathbf{d}_{k-1} = (z(t_1), z(t_2), \ldots, z(t_{k-1}))$. That is,

$$\pi_k(\theta_z|\mathbf{d}_{k-1}, \triangle z_k) \propto \frac{\lambda_z^{\alpha_z \triangle t_k}}{\Gamma(\alpha_z \triangle t_k)} (\triangle z_k)^{\alpha_z \triangle t_k - 1} e^{-\alpha_z \triangle z_k} \pi_k(\theta_z|\Lambda_k),$$

where $\pi_k(\theta_z|\Lambda_k)$ is the prior with hyperparameters Λ_k=($\mu_a^{(k)}$, $\mu_b^{(k)}$, $\sigma_a^{2(k)}$, $\sigma_b^{2(k)}$) updated through $\mathbf{d}_{k-1}$. The hyperparameters of prior π_{k+1} are updated by matching the prior moments (or quantiles) with the posterior sample moments (or quantiles) based on the posterior sample of θ_z at time t_k, namely $\mathbf{a}^{(k)} = (a_1^{(k)}, \ldots, a_M^{(k)})$, $\mathbf{b}^{(k)} = (b_1^{(k)}, \ldots, b_M^{(k)})$ and $\lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_M^{(k)})$ via the MCMC procedure. For example, $\mu_a^{(k+1)} = \bar{a}^{(k)} = \sum_{l=1}^{M} a_l^{(k)}/M$ and $\sigma_a^{2(k)} = \sum_{l=1}^{M}(a_l^{(k)} - \bar{a}^{(k)})^2/(M-1)$, and the rest follows.
Furthermore, the predictive inference on the lifetime distribution can also be updated sequentially in time. As more data are observed, more accurate inference can be made. Therefore, the termination time of the experiment can be decided by the time when the predictive inference meets the pre-specified requirement such as the length of the interval estimates or a tolerance bound of the standard error of the quantity of interest.

4. Section 4

In this section, we analyze a simulated data set under a two-level SSADT gamma degradation process from the proposed model with true parameters $(a, b, \lambda) = (3, -2000, 10)$, sample size $n = 12$, stress levels $S_1 = 50$ and $S_2 = 100$, and the threshold value $Y_f = 2$. Consider $\Delta t_k = 10k$, $k = 1, 2, \cdots, 25$, and $\zeta_1 = 15, \zeta_2 = 25$,

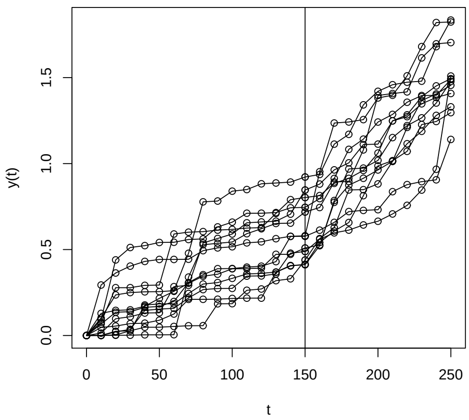

Figure 1: Illustrative Example

i.e. the stress levels are changed at $t = 150$. Figure 1 shows the simulated SSADT gamma process and Figure 2 is the data of a new product under use condition $S_0 = 25$. In the Bayesian analysis, the hyperparameters

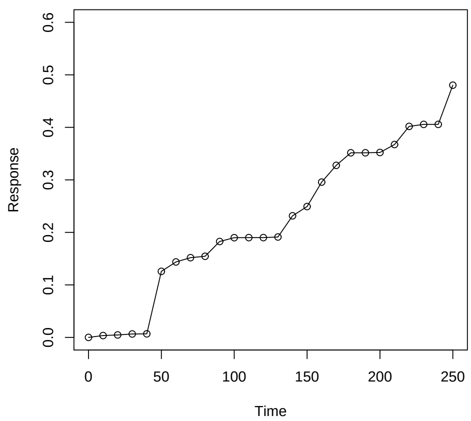

Figure 2: Degradation data of a new product.

are $(\mu_a, \mu_b, \sigma_a^2, \sigma_b^2) = (3.5, -1930, 10^2, 200^2)$, and $c = 0, d = 0$ so that $\pi(\lambda) \propto \frac{1}{\lambda}$, the natural noninformative prior. An MCMC procedure was performed by burning the first 20000 iterations, and taking one sample in every 10 iterations to get 1000 posterior samples. Figures 3 shows the sequential step by step prediction of the MTTF and $t_{0.9}$, respectively, under normal use condition along with their 95% credible intervals.

5. Conclusions

In this article, a Bayesian approach is applied to an SSADT test under gamma processes. We form the prior distribution based on previous SSADT data for developing degradation test of a new product under normal use condition, and update the prior distributions once more data are collected. In addition, the experimental time can be determined by specifying the desired estimation accuracy. Different criteria such as the cost consideration can be incorporated to decide the experimental time.

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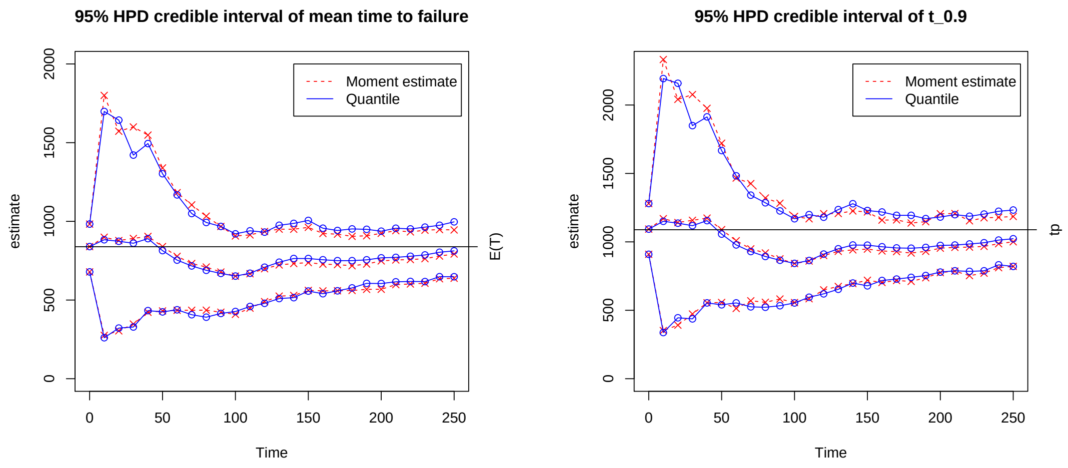

Figure 3: 95% HPD credible intervals of $E(T|S_0)$ and $t_{0.9}(S_0)$.

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