# Geostatistical Mixed Beta Regression: A Bayesian approach

Jorge I. Figueroa-Zúñiga*
Universidad de Concepción, Concepción, Chile - jifiguer@gmail.com

Bernardo M. Lagos-Alvarez
Universidad de Concepción, Concepción, Chile - bla@udec.cl

Roberto Fustos Toribio
Universidad de Chile, Santiago, Chile - rfustos@alges.cl

Francisco Torres-Avilés
Universidad de Santiago de Chile, Santiago, Chile - francisco.torres@usach.cl

**Abstract**

This paper is based on recent research that focuses on regression modeling for geostatistical bounded data, with emphasis in proportions measured on a continuous scale. Specifically, it deals with beta regression models with a mixed effect in order to control the spatial variability from a Bayesian approach. Until now, various studies have been performed for geostatistical models, primarily restricted to response variable with support in the real line. However, variables with limited-range are common in practice, and is necessary have a model with bounded support, and flexible in the sense that it can model asymmetries.
We use a suitable parameterization of the beta distribution in terms of its mean and the precision parameter, which allows for both parameters to be modeled through regression structures that may involve fixed and random effects. Specification of prior distributions is discussed, computational implementation via Gibbs sampling is provided, finally it is illustrated using simulated data.

**Keywords**: Bayesian analysis; Beta distribution; Mixed regression models; Geostatistical models.

## 1. Introduction

Geostatistics is the area of the spatial statistics that pretends to primarily model response variables measured over continuous regions, and generally, normal or at least symmetrical distributions are assumed. This assumption is often forced from the non-linear transformation of the response. In many situations appear geostatistical non-Gaussian data, especially data with bounded support, for instance those quantities that measures concentrations and proportions. Specifically we considered those responses measured on a continuous scale, which are restricted to the unit interval $(0, 1)$. Situations where the response, say $y$, is limited to a known interval $(a, b)$ is also accomodated through the linear transformation $y^* = (y - a)/(b - a)$. The response variable is assumed to be beta distributed with mean (and possibly a precision parameter) modeled using fixed and random effects. The substantial advantage to consider a beta modeling is due to the flexibility that it provides. In fact, the beta family includes left or right skewed, symmetric, J-shaped, and inverted J-shaped distributions. This article concerns the latter case, where interpolation is needed to predict values at unsampled sites.
Diggle, Tawn, and Moyeed (1998) employed spatial generalized linear mixed models (GLMMs) for spatially dependent non-Gaussian variables observed in a continuous region and considered the minimum mean-squared error (MMSE) prediction under the Bayesian framework. Mixed effects models have provided a convenient means of modeling spatial correlations by using random effects, with common spatial correlation structures including, for example, random fields for geostatistical data, and autoregressive structure for areal data (Yasui and Lele, 1997; Waller, et al., 1997).
In the present work, we will also use a spatial GLMMs to model spatial non-Gaussian data. The shared features of data arising from these studies fall into the geostatistical type, where the outcomes are random variables indexed by locations that vary continuously over a subset of an Euclidean space.
Following Figueroa-Zúñiga et al. (2013), our proposed model uses a parameterization of the beta law in terms of its mean and an additional positive parameter that can be regarded as a precision parameter. The mean of the response

variable is conveniently linked with a GLMMs structure by the logit link function. It assumes that the precision parameter is not constant over the observations, but rather it is related to a mixed effects function through a log link. To formulate our proposed models, we adopt a Bayesian approach. We address the issues of model fitting via Gibbs sampling, choice of prior distributions, and model selection based on the deviance information criterion. Simulated data analysis are presented for illustration.

**2. Bayesian beta geostatistical model**

Due to the flexibility of the beta distribution in terms of the variety of density shapes that can be accommodated, this distribution is a natural choice for modeling continuous data that are restricted to the interval $(0, 1)$. The probability density function of a variable $y$ following a beta distribution parameterized in terms of its mean $\mu$ and a precision parameter $\psi$ is given by

$$f(y|\mu, \psi) = \frac{\Gamma(\psi)}{\Gamma(\mu\psi)\Gamma((1-\mu)\psi)} y^{\mu\psi-1}(1-y)^{(1-\mu)\psi-1}, \quad 0 < y < 1, \tag{1}$$

where $\Gamma(\cdot)$ denotes the gamma function, $\mu = \mathrm{E}(y)$ $(0 < \mu < 1)$, and $\psi > 0$ can be interpreted as a precision parameter, since $\mathrm{Var}(y) = \mu(1-\mu)/(1+\psi)$ and, hence, for each fixed value of the mean $\mu$, $\psi$ is inversely proportional to the variance of $y$. If $y$ has density function (1), we write $y \sim \mathrm{beta}(\mu\psi, (1-\mu)\psi)$.
Now, let $y_1, \ldots, y_n$ be $n$ random variables ($y_i = y(x_i)$, with $x_i \in D \subset \mathbb{R}^d$) such that $y_i \sim \mathrm{beta}(\mu_i\psi_i, (1-\mu_i)\psi_i)$. The definition of a beta spatial model requires a transformation of the mean $\mu_i$ of $y_i$, $i = 1, \ldots, n$, that maps the interval $(0, 1)$ onto the real line. A convenient and popular link function is the logit link. It is then assumed that $\ln\{\mu_i/(1-\mu_i)\} = r_i^\top \beta_\mu$, where $r_i$ is a vector of known covariates for the $i$-th location and $\beta_\mu$ denotes a vector of regression coefficients. The first element of $r_i$ is usually taken as 1 to allow an intercept.
The beta regresion model described above does not involve spatially correlated random effects, inherent in the data. Extending previous work on Bayesian generalized linear models of Dey et al. (2000) and Bayesian beta regression Figueroa-Zúñiga et al. (2013), we define below the beta geostatistical regresion model that includes spatially correlated random effects on the location and precision parameter.
Let $y_1, \ldots, y_n$ be $n$ continuos data that exhibit a spatial correlation on Euclidian space $\mathbb{R}^2$, where for which each of its components, $y_i$, takes values on the interval $(0, 1)$. Consider also a regression model with the following structure:

$$G(\mathrm{E}(y_i|\Theta)) = r_i^\top \beta_\mu + z_i, \tag{2}$$

$i = 1, \ldots, n$, where $G(\cdot)$ is a vector-function linking the conditional mean response vector $\mathrm{E}(y_i|\Theta)$ with the linear mixed model $\eta_i = r_i^\top \beta_\mu + z_i$, for which $r_i$ is the design matrix of dimension $1 \times p$ corresponding to the vector $\beta_\mu$ of regression coefficients (the fixed effects) and $z_i$ are unobserved spatially correlated random effects indexed by the localizations $i$'s. In practice, we assume an isotropic stationary random field structure for the $z_i$'s, with validate covariance structure $\mathcal{C}(\cdot, \Theta)$ and $\Theta$ is the vector of all parameters of interest involved in the model.
Furthermore, we propose to model the precision parameter $\psi$ in two ways, which define the models 1 and 2. The first way is to set the parameter $\psi$ as a constant, so it will be necessary to assign a prior distribution. The second way, as suggested by Smithson and Verkuilen (2006), making it possible to assign a structure regression as in the case of location parameter $\mu$ as follows

$$H(\psi_i) = \zeta_i = w_i^\top \beta_\psi$$

where $H(\cdot)$ corresponds to an appropriate link function according to the nature of the parameter $\psi$, $w_i^\top = (w_{i1}, \ldots, w_{iq})$ is the design vector corresponding to a $q \times 1$ vector associate to fixed effects. Note that, the design matrices $w = (w_1, \ldots, w_n)^\top$ may, but are not required to, contain the same predictor variables as the matrices $r = (r_1, \ldots, r_n)^\top$, ie, $q \le p$, and $\beta_\psi$ is the vector of parameters associated with these covariates.
In mixed models, the random effects $\mathbf{z}_n = \{z_1, \ldots, z_n\}$ are typically assumed to be multivariate normally distributed, namely $\mathbf{z}_n|\Sigma_z \sim N_n(0, \Sigma_z)$ where $\Sigma_z := \Sigma_z(\Theta_z)$ must come from a theoretical variogram model $\gamma_z(\cdot, \Theta_z)$ or the covariogram structure $\mathcal{C}_z(\cdot, \Theta_z)$. A commonly used parameterization for the covariance $\Sigma_z = (\sigma_{ij})$ is $\sigma_{ij} = \sigma^2\rho(\phi, \kappa; h_{ij})$ where $\sigma^2$ is the variance of the spatial process and $\rho(\phi, \kappa; h_{ij})$ a valid correlation function with a scale parameter $\phi$, which controls the decay rate of the correlation between the locations $s_i$ and $s_j$, while the distance $h_{ij}$ increases. For this parametrization, the parameter vector $\Theta_z$ contains three additional parameters $\sigma$, $\phi$ and $\kappa$. In most applications a monotonic correlation function is chosen to be the exponential function which has the form

$\rho(\phi, \kappa; h_{ij}) = \exp(-\{h_{ij}/\phi\}^{\kappa})$ with $0 < \kappa < 2$. Ecker and Gelfand (1997) propose several other parametric correlation forms, such as the Gaussian, Cauchy, spherical and the Bessel. Alternative specifications for the variograms are reviewed in Banerjee et al. (2004) and Cressie and Wikle (2011).
A separate set of location-specific random effects, $\boldsymbol{U} = (U_1, \ldots, U_n)^{\top}$ is often suggested in equation (2) to account for unexplained non-spatial variation (Diggle et al., 1998), where $U_i, 1, \ldots, n$ are considered to be independent, arising from a normal distribution, $\boldsymbol{U}|\tau^2 \sim N(0, \tau^2 I)$. The $\tau^2$ is known in geostatistics as the nugget effect and introduces a discontinuity at the origin of the covariance function, $\mathcal{C}_z(\cdot, \Theta)$, so $\sigma_{ij} = \tau^2 + \sigma^2\rho(\phi, \kappa; h_{ij})$.
So our basic model assume that

$$y_i|\Theta \overset{ind.}{\sim} \text{beta}(\mu_i\psi_i, (1-\mu_i)\psi_i), \text{ for } i = 1, 2, \ldots, n \tag{3}$$

i.e., conditionally on $\Theta = (\beta, \Theta_z)$, $\beta = (\beta_\mu, \beta_\psi)$, the $y_i$'s are independent and have reparametrize beta probability density function given by (1).
In order to complete the specification of the beta geostatistical mixed models described above, elicitation of prior distributions for all unknown parameters is required. Multivariate normal prior distributions are typically considered for the fixed effects, i.e., $\beta_\mu \sim N_p(\mu_{\beta_\mu}, \Sigma_{\beta_\mu})$ and $\beta_\psi \sim N_q(\mu_{\beta_\psi}, \Sigma_{\beta_\psi})$. Vague priors are usually specified by taking large values for the prior variances. However, the impact of the scale choice under the normal model cannot be neglected. An alternative strategy is to consider a multivariate $t$-distribution, i.e., $\beta_\mu \sim t_p(v, \mu_{\beta_\mu}, \Sigma_{\beta_\mu})$ and $\beta_\psi \sim t_q(v, \mu_{\beta_\psi}, \Sigma_{\beta_\psi})$, and to specify an appropriated value for $\nu_\beta$, the degrees of freedom parameter. We have chosen the latter alternative.
In our work, we will use priors already suggested and tested by the literature. For parameters that control their spatial effects, ie, $\sigma_z^2$ and $\phi$, we use as priors, the inverse gamma and uniform distributions respectively. With respect to the $\kappa$ parameter controls the curvature of the correlations between the locations we will use a (0, 2) uniform distribution as prior. For model 1, we also provide a prior distribution for the parameter $\psi$, so we assume a lognormal distribution for the dispersion parameter $\psi$ with fixed hyperparameters.

**3 Model fitting using Markov chain Monte Carlo sampling.**
Let $y^{\top} = (y_1, ..., y_n)$, $\eta^{\top} = (\eta_1, ..., \eta_n)$ and $\zeta^{\top} = (\zeta_1, ..., \zeta_n)$, where $\eta_i = r_i^{\top}\beta_\mu + z_i$ and $\zeta_i = w_i^{\top}\beta_\psi$. We note that on the assumptions above, the $\eta_i$'s conditioned by $\beta_\mu$, $\psi$ (model 1) or $\beta_\mu$, $\beta_\psi$ (model 2), $\sigma_z^2$, $\phi$ and $\kappa$, are dependents through spatial random effects $z_i$ and have density function $f(\eta_i|\beta_\mu, \sigma_z^2, \phi, \kappa) \propto f(z_i|\beta_\mu, \sigma_z^2, \phi, \kappa)$, $i = 1, ..., n$. And considering that $\beta_\mu, \psi, \sigma_z^2, \phi$ y $\kappa$ are independent, the joint posterior density $f(\beta_\mu, \psi, \sigma_z^2, \phi, \kappa, \eta|y)$, in the case of model 1, is proportional to

$$\left[\prod_{i=1}^{n} f(y_i|\eta_i, \psi)\right] f(\eta|\beta_\mu, \sigma_z^2, \phi, \kappa) f(\psi) f(\beta_\mu) f(\sigma_z^2) f(\phi) f(\kappa)$$

and in the case of model 2, the joint posterior density $f(\beta_\mu, \beta_\psi, \sigma_z^2, \phi, \kappa, \eta|y)$ is proportional to

$$\left[\prod_{i=1}^{n} f(y_i|\eta_i, \beta_\psi)\right] f(\eta|\beta_\mu, \sigma_z^2, \phi, \kappa) f(\beta_\mu) f(\beta_\psi) f(\sigma_z^2) f(\phi) f(\kappa).$$

Gibbs sampling can be used to generate a sample from the joint posterior densities, $f(\beta_\mu, \psi, \sigma_z^2, \phi, \kappa, \eta|y)$ for model 1 and $f(\beta_\mu, \beta_\psi, \sigma_z^2, \phi, \kappa, \eta|y)$ for model 2. In this context, Gibbs sampling involves an iterative sampling from full conditional distributions, for the model 1 we have:

$$\begin{gathered} f(\beta_\mu|\psi, \sigma_z^2, \phi, \kappa, \eta, y), \quad f(\psi|\beta_\mu, \sigma_z^2, \phi, \kappa, \eta, y), \quad f(\sigma_z^2|\beta_\mu, \psi, \phi, \kappa, \eta, y), \\ f(\phi|\beta_\mu, \psi, \sigma_z^2, \kappa, \eta, y), \quad f(\kappa|\beta_\mu, \psi, \sigma_z^2, \phi, \eta, y), \text{ and} \\ f(\eta_i|\eta_j, \beta_\mu, \psi, \sigma_z^2, \phi, \kappa, y), \quad i, j = 1, ...n, i \neq j, \end{gathered}$$

and for the model 2 we have:

$$\begin{gathered} f(\beta_\mu|\beta_\psi, \sigma_z^2, \phi, \kappa, \eta, y), \quad f(\beta_\psi|\beta_\mu, \sigma_z^2, \phi, \kappa, \eta, y), \quad f(\sigma_z^2|\beta_\mu, \beta_\psi, \phi, \kappa, \eta, y) \\ f(\phi|\beta_\mu, \beta_\psi, \sigma_z^2, \kappa, \eta, y), \quad f(\kappa|\beta_\mu, \beta_\psi, \sigma_z^2, \phi, \eta, y), \text{ and} \\ f(\eta_i|\eta_j, \beta, \psi, \sigma_z^2, \phi, \kappa, y), \quad i, j = 1, ...n, i \neq j. \end{gathered}$$

The inference is performed from the posterior distributions of the parameters, more specifically, the interest is on average. Using OpenBUGS is possible to obtain the results.

**4. Illustration via simulations**

To perform the simulation study, it is necessary to have data from the above process. For this reason, and following the techniques provided by Diggle and Ribeiro (2007), we proceed to simulate 100 random fields with beta distribution, each with $n = 30$ observations distributed irregularly, corresponding to the measurement of the random variable in each locality and covariates involved with the regression parameters for both models and parameters for spatial dependence. In the case of model 1, we have:

$$y_i|\beta_\mu, \psi, \sigma_z^2, \phi, \kappa \;\sim\; beta(\mu_i\psi, (1-\mu_i)\psi)$$

with

$$log\left\{\frac{\mu_i}{1-\mu_i}\right\} = \eta_i = r_i^\top \beta_\mu + z_i$$

$\beta_\mu = (4, 2)^\top$ and $\mathbf{z} = (z_1, ..., z_n)^\top \sim N_n(0, \Sigma_z)$, $\Sigma_z = (\sigma_{ij})$, $\sigma_{ij} = \sigma_z^2\rho(\phi, h_{ij})$, with the function $\rho$ associated to a power-exponential model, $\sigma_z^2 = 1$, $\phi = 20$, $\kappa = 1.5$, and $\psi = 10$, $i = 1, ..., n$.

For the model 2, we have:

$$y_i|\beta_\mu, \beta_\psi, \sigma_z^2, \phi, \kappa \;\sim\; beta(\mu_i\psi_i, (1-\mu_i)\psi_i)$$

with

$$log\left\{\frac{\mu_i}{1-\mu_i}\right\} = \eta_i = r_i^\top \beta_\mu + z_i \quad ; \quad log\{\psi_i\} = \zeta_i = w_i^\top \beta_\psi,$$

$\beta_\psi = (2.3, 0.01)^\top$ and the other parameters as in model 1.
Table 1 shows the proposed models with different priors assignments for each parameter.

**Table 1:** Prior distributions for the spatial beta model.

| Model | | | Priors |
|---|---|---|---|
| model 1a | $\beta_\mu \sim t_p(v_1, \mu_{\beta_\mu}, \Sigma_{\beta_\mu})$ | and | $\psi \sim LN(4, 1)$ |
| model 1b | $\beta_\mu \sim t_p(v_2, \mu_{\beta_\mu}, \Sigma_{\beta_\mu})$ | and | $\psi \sim LN(4, 1)$ |
| Model | | | Prior for $\beta_\mu$ and for $\beta_\psi$ |
| model 2a | $\beta_\mu \sim t_p(v_1, \mu_{\beta_\mu}, \Sigma_{\beta_\mu})$ | and | $\beta_\psi \sim t_q(v_1, \mu_{\beta_\psi}, \Sigma_{\beta_\psi})$ |
| model 2b | $\beta_\mu \sim t_p(v_2, \mu_{\beta_\mu}, \Sigma_{\beta_\mu})$ | and | $\beta_\psi \sim t_q(v_2, \mu_{\beta_\psi}, \Sigma_{\beta_\psi})$ |

For the parameters $\sigma_z^2$ and $\phi$, we will use the priori distributions inverse gamma $(IG(0.01, 0.01))$ and uniform $(U(\xi_{inf}, \xi_{sup}))$, respectively. The values $\xi_{inf}$ and $\xi_{sup}$ depend on the maximum and minimum distance between locations and can be obtained in using the following relations,

$$\xi_{inf} = \frac{(-log 0.01)^{1/1.5}}{\text{maximum distance}} \qquad \text{y} \qquad \xi_{sup} = \frac{(-log 0.01)^{1/1.5}}{\text{minimum distance}}$$

discussed in the tutorial of OpenBUGS software. The hyperparameters $\mu_{\beta_\mu}$, $\mu_{\beta_\psi}$ are represented by the vector $(0, 0)^\top$, the hyperparameters $\Sigma_{\beta_\mu}$, $\Sigma_{\beta_\psi}$ are represented by the diagonal matrix $(0.01^{-1}, 0.01^{-1})$, while the degrees of freedom are $v_1 = 5$ and $v_2 = 10$ respectively. Note in the model 1, that if $\psi \sim LN(\mu_\psi, \sigma_\psi^2)$ with $\mu_\psi = 4$ and $\sigma_\psi^2 = 1$, we have $E(\psi) = 90$ and $\text{Var}(\psi) = 13923.38$. This values for the prior has been carefully chosen, because in practice, the parameter $\psi$ strangely takes small values, because of this, it would be inappropriate assign a prior with $\mu_\psi = 0$. Finally as the parameter $\kappa$ moves between 0 up to 2 we will use the uniform distribution $(0, 2)$.
A sensitivity analysis was carried out for each specified model in the Table 1 considering three goodness of fit measures; Deviance Information Criterion (DIC) proposed by Spiegelhalter et al. (2002), the Expected Akaike Information Criterion (EAIC) introduced by Brooks (2002) and Expected Bayesian Information Criterion (EBIC) developed by Carlin and Louis (2001). All the numerical results presented in this paper were obtained using OpenBUGS by considering 100,000 Monte Carlo iterations and discarding the first 20,000 as burn-in. In Table 2, we observe that both models 1 and 2 presents a best fit for priors with less degrees of freedom.

**Table 2:** Goodness of fit criteria for each model from Table 1.

| Model | DIC | EAIC | EBIC |
|---|---|---|---|
| model 1a | 40.63 | 48.94 | 97.70 |
| model 1b | 41.35 | 48.95 | 97.71 |
| model 2a | 38.22 | 50.96 | 107.80 |
| model 2b | 42.20 | 51.32 | 108.20 |

Tables 3 and 4 present the results summarized using the bias and the square root of the mean squared error (MSE) for each estimator of the parameters of the 100 samples under different models considered in Table 1. These results show that the parameters estimated under the Bayesian methodology are similar to the corresponding true values of the parameters.

**Table 3:** Summary based on 100 simulations of model 1.

| Model | | $\beta_\mu(1)$ | $\beta_\mu(2)$ | $\psi$ | $\sigma^2_{\mathbf{z}}$ | $\phi$ | $\kappa$ |
|---|---|---|---|---|---|---|---|
| model 1a | $RealBias$ | 0.00 | 0.00 | 0.16 | 0.07 | -0.09 | -0.08 |
| | $\sqrt{MSE}$ | 0.53 | 0.25 | 1.35 | 0.43 | 1.74 | 0.25 |
| model 1b | $RealBias$ | 0.00 | 0.00 | 0.17 | 0.07 | -0.09 | -0.08 |
| | $\sqrt{MSE}$ | 0.53 | 0.26 | 1.37 | 0.43 | 1.72 | 0.25 |

**Table 4:** Summary based on 100 simulations of model 2.

| Model | | $\beta_\mu(1)$ | $\beta_\mu(2)$ | $\beta_\psi(1)$ | $\beta_\psi(2)$ | $\sigma^2_{\mathbf{z}}$ | $\phi$ | $\kappa$ |
|---|---|---|---|---|---|---|---|---|
| model 2a | $RealBias$ | 0.00 | 0.00 | 0.04 | 0.78 | 0.07 | -0.09 | -0.08 |
| | $\sqrt{MSE}$ | 0.54 | 0.26 | 1.39 | 0.66 | 0.43 | 1.72 | 0.25 |
| model 2b | $RealBias$ | 0.00 | -0.00 | 0.03 | 1.30 | 0.07 | -0.09 | -0.08 |
| | $\sqrt{MSE}$ | 0.54 | 0.27 | 1.39 | 0.77 | 0.43 | 1.72 | 0.25 |

In addition, the necessary diagnostic tests (such as convergence, autocorrelation, history) were performed, from which desirable behaviors were observed in the chains. The multivariate version of Gelman and Rubin's convergence diagnostic proposed by Gelman (1992) indicates that the chain is convergent because the multivariate proportional scale reduction factor (mprf) equals $1.02$ ($< 1.2$). Also, for each parameter, we checked that the convergence is achieved for each chain.

**5. Conclusions**

Beta regression modeling has gained increasing popularity after the work proposed by Ferrari and Cribari-Neto (2004), who described a beta regression model parameterized in terms of the mean response and a common precision parameter, and developed frequentist inference and basic diagnostic tools for the proposed model. A complementary approach proposed by Smithson and Verkuilen (2006) considers that the precision parameter is not fixed, that is, this parameter was modeled in a regression manner. A Bayesian beta regression model was studied by Branscum et al. (2007). After Figueroa-Zúñiga et al. (2013) extended these ideas for a mixed beta regression model under a Bayesian perspective.

The present paper considered the Bayesian inference for the geostatistical beta regression based on two different approaches. First, the precision parameter was assumed to be fixed, i.e., the same for all observations. A linear regression structure was also proposed for the mean parameter through a logit link function. Our results are readily extended to other link choices.

Specification of different priors for the common precision parameter was studied. We considered a prior distribution for $\psi$ of the type lognormal. Second, the precision parameter was modeled through its own linear regression structure using a log link. For the mean submodel is considered a mixed-effects model with a multivariate Student-$t$ distribution for the fixed effect, for the spatial random effect we considered a Gaussian process, and for the precision submodel only a fixed effect model with a multivariate Student-$t$ distribution was considered. Incorporating spatial random effects to

the precision the interpretation becomes complex. We must also consider when making adjustments with the random effect, the results with a high cost computational do not produce significant difference to report them. Our empirical applications yielded good results in terms of model fit and diagnostics tests. It is worth mentioning that in this context, it is necessary to perform a careful model selection for the precision modeling including more or fewer fixed effects since it is not clear in advance which model is more plausible.
An advantage of this approach is the easy implementation for the imputation of missing data Carrigan et al., (2007), a common situation in practice.

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