An objective Bayesian approach to tolerance intervals for the Poisson distribution

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Abstract

In this paper we will have a look at Bayesian tolerance intervals for the Poisson distribution. Tolerance intervals could be of interest in quality control. A tolerance interval gives information about a certain proportion or more of the population, with a given confidence level. This proportion is also referred to as the content of a tolerance interval. Whereas a confidence interval gives information about an unknown parameter. The goal of a tolerance interval is to contain at least a specified proportion of the population with a specified degree of confidence. An objective Bayesian approach will be used. The coverage rates obtained for one-sided and two-sided intervals were relatively good, where the coverage rates were most of the time at or above 0.95.

Keywords: Noninformative priors; posterior; predictive density.

1. Introduction

In this paper we will focus on Bayesian tolerance intervals for the Poisson distribution. Tolerance intervals could be of interest in quality control. Wang & Tsung (2009) state the following: "The construction of tolerance intervals to measure discrete quality characteristics has been one of the major tasks in developing quality control systems used in manufacturing and pharmaceutical sectors". A tolerance interval gives information about a certain proportion or more of the population, with a given confidence level. This proportion is also referred to as the content of a tolerance interval. Whereas a confidence interval gives information about an unknown parameter. The goal of a tolerance interval is to contain at least a specified proportion of the population with a specified degree of confidence. In this paper π will indicate the content, and $1-\alpha$ the confidence level. There are three main types of tolerance intervals: the $(\pi, 1-\alpha)$ tolerance interval, where π is the content and $1-\alpha$ the confidence level; the π-expectation tolerance interval, where π is the expected coverage of the interval. The π - expectation intervals focus on the prediction of one or a few future observations from the process; the fixed-in-advance tolerance interval, where the interval is constant and one wishes to estimate the proportion of process measurements it contains.

The above mentioned intervals can be one-sided or two-sided tolerance intervals. The two-sided interval can take on two different types. From Krishnamoorthy & Mathew (2009) the one is constructed such that it would contain at least a proportion π of the population with confidence level $1-\alpha$, and the other type is constructed such that it would contain at least a proportion π of the centre of the population with confidence level $1-\alpha$. The latter is referred to as an equal-tailed tolerance interval. Our interest is to investigate Bayesian tolerance limits and intervals for the Poisson distribution.

2. Prior and Posterior

When we specify Bayesian models, we have to decide on prior distributions for unknown parameters. We will investigate a number of noninformative priors. A noninformative prior is used when little or no prior information is available.

The Jeffreys' prior: Jeffreys (1961) argued that if there is no prior information about the unknown parameter, then there is also no information about any one-to-one transformation of the parameter, and therefore the rule for determining a prior should give a similar result if it is applied to the transformed parameter. The Jeffreys prior is proportional to the square root of the determinant of the Fisher information matrix and is given by

$$\pi_J(\lambda) \quad \propto \quad \lambda^{-\frac{1}{2}}. \tag{1}$$

The reference prior: The reference prior was introduced by Bernardo (1979) and Berger & Bernardo (1992). As mentioned by Pearn & Wu (2005) the reference prior maximises the difference in information about the parameter provided by the prior and posterior. The reference prior is derived in such a way that it provides as little information as possible about the parameter. As in the case of the Jeffreys prior, the reference prior method is derived from the Fisher information matrix. From Yang & Berger (1997) the reference prior is the same as the Jeffreys' prior for this case and is thus given by

$$\pi_R(\lambda) \quad \propto \quad \lambda^{-\frac{1}{2}}. \tag{2}$$

The probability matching prior: A probability matching prior is a prior distribution under which the posterior probabilities match their coverage probabilities. The fact that the resulting Bayesian posterior intervals of level $1-\alpha$ are also good frequentist confidence intervals at the same level is a very desirable situation. Datta & Ghosh (1995) derived the differential equation which a prior must satisfy if the posterior probability of a one sided credibility interval for a parametric function and its frequentist probability agree up to $O\left(n^{-1}\right)$ where n is the sample size. The probability matching prior for a linear combimation of k Poisson rates, $\sum_{i=1}^{k} a_i\lambda_i$, is given by $\pi_{PM}(\underline{\lambda}) \propto \left(\sum_{i=1}^{k} a_i^2\lambda_i\right)^{\frac{1}{2}} \prod_{i=1}^{k}\lambda_i^{-1}$. See Raubenheimer & Van der Merwe (2014) for further discussion. If we let $k=1$ and $a=1$, the probability matching prior is given by

$$\pi_{PM}(\lambda) \quad \propto \quad \lambda^{-\frac{1}{2}}. \tag{3}$$

From the above it is clear that the Jeffreys, reference and probability matching priors yield the same prior. Using this prior, the posterior distribution will be

$$\pi(\lambda\,|data) \quad \propto \quad e^{-\lambda}\lambda^{x-\frac{1}{2}} \qquad\qquad \lambda > 0. \tag{4}$$

The predictive density for a future observation, x_f, will then be

$$\begin{aligned} f(x_f\,|data) &= \int_0^\infty f(x_f\,|\lambda)\,\pi(\lambda\,|x)\,d\lambda \\ &= \frac{\Gamma\left(x_f+x+\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)x_f!}\frac{1}{2^{x_f+x+\frac{1}{2}}} \qquad\qquad x_f = 0,1,\ldots,\infty \end{aligned}$$

This predictive density can be used to construct the second type of tolerance interval mentioned in Section 1, the $\pi-$expectation tolerance interval. As mentioned, the $\pi-$expectation tolerance intervals focus on the prediction of one or a few future observations from the process. The posterior distribution given in Equation 4 can now be used to obtain the required tolerance interval.

3. Illustration

Consider the following study for the interval estimation of the 95^{th} percentile of the Poisson distribution. For given x, $x = 0,1,\ldots 50$, 10 000 values are simulated from the posterior distribution of λ. For each value of λ, the corresponding 95^{th} percentile (P_{95}) of the Poisson distribution is found. From the sorted 10 000 values of P_{95} the lower limit (2.5%) and upper limit (97.5%) is found in the case of the two-sided interval. For given λ, $\lambda = 1,\ldots,15$, the probabilities for all values of x which yielded an interval which contains the true λ are added to obtain the coverage probability. The results are given in Table 1 and plotted in Figures 1 and 2.

Table 1: Interval estimation of the 95^{th} percentile of the Poisson distribution.

λ	P_{95}	$P(X \leq P_{95})$	**Two-sided coverage**	**One-sided coverage**
1.0	3	0.9810	0.9963	1.0000
1.5	4	0.9814	0.9955	1.0000
2.0	5	0.9834	0.9955	1.0000
2.5	5	0.9580	0.9858	1.0000
3.0	6	0.9665	0.9383	0.9502
3.5	7	0.9733	0.9599	0.9698
4.0	8	0.9786	0.9736	0.9084
4.5	8	0.9597	0.9718	0.9389
5.0	9	0.9682	0.9796	0.9596
5.5	10	0.9747	0.9624	0.9734
6.0	10	0.9574	0.9626	0.9826
6.5	11	0.9661	0.9727	0.9570
7.0	12	0.9730	0.9576	0.9704
7.5	12	0.9573	0.9582	0.9797
8.0	13	0.9658	0.9690	0.9576
8.5	14	0.9726	0.9561	0.9256
9.0	14	0.9585	0.9567	0.9450
9.5	15	0.9665	0.9674	0.9597
10.0	15	0.9513	0.9626	0.9707
10.5	16	0.9604	0.9570	0.9496
11.0	17	0.9678	0.9672	0.9625
11.5	17	0.9542	0.9631	0.9723
12.0	18	0.9626	0.9584	0.9542
12.5	19	0.9694	0.9481	0.9654
13.0	19	0.9573	0.9491	0.9741
13.5	20	0.9649	0.9693	0.9585
14.0	20	0.9521	0.9691	0.9684
14.5	21	0.9604	0.9524	0.9516
15.0	21	0.9673	0.9625	0.9626

From Table 1 and Figures 1 and 2 it can be seen that the coverage rates are most of the time at or above the nominal value of 0.95. When $\lambda = 3$, 12.5 and 13 the coverage rates are below 0.95 for the two-sided case, and when $\lambda = 4$, 4.5, 8.5, 9 and 10.5 the coverage rates are below 0.95 for the one-sided case.

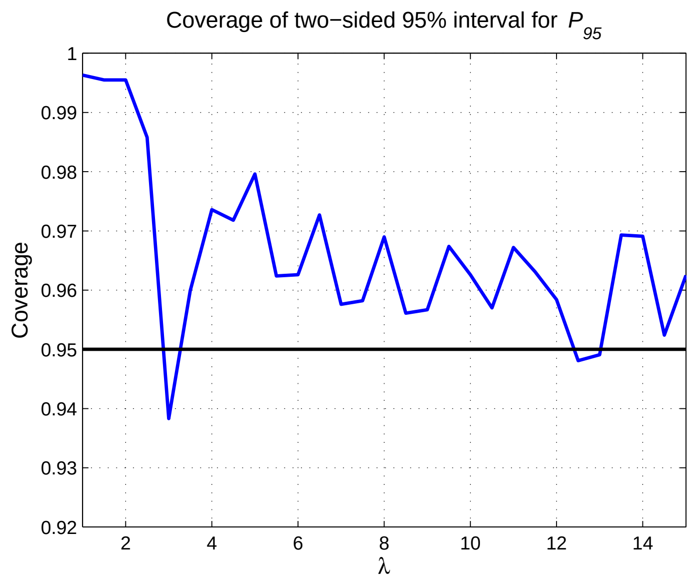

Figure 1: Coverage of two-sided 95% interval for P_{95}.

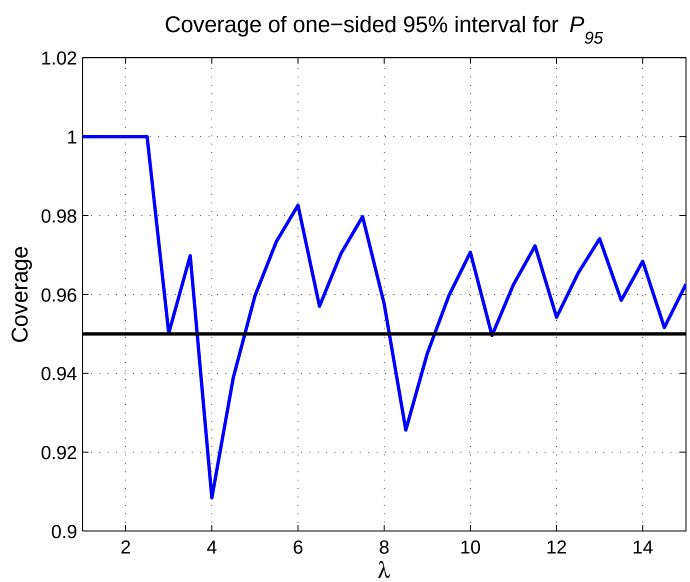

Figure 2: Coverage of one-sided 95% interval for P_{95}.

Consider the following example from Montgomery (1996), Example 6-3 on page 277. This example deals with the number of nonconformities observed in 26 successive samples of 100 printed circuit boards. The inspection unit is defined as 100 boards. The 26 samples contained 516 nonconformities, and λ is estimated by $\overline{\lambda} = 19.85$. It was found that units 6 and 20 were out-of-control, and they were eliminated after further investigation. Revised limits were calculated using the remaining samples, with $m = 24$ and $\sum_{i=1}^{m} x_i = 472$. The average number of nonconformities per inspection unit was recalculated as $\overline{\lambda} = 19.67$. Considering this example, we have $m = 24$ and $\sum_{i=1}^{m} x_i = 472$. Figure 3 shows the posterior distribution of λ.

Summary statistics for the posterior distribution of λ:

mean = 19.69; standard deviation = 0.9057; median = 19.67; 95% credibility interval = (17.95; 21.50).

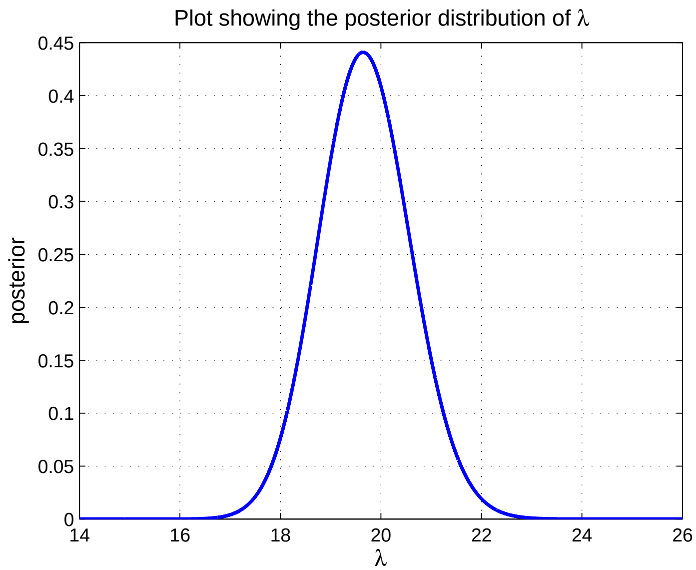

Figure 3: Posterior distribution of λ, when $m = 24$ and $\sum_{i=1}^{m} x_i = 472$.

We construct the $(\pi, 1-\alpha)$ tolerance interval for this example, where $\pi = 0.95$ and $1-\alpha = 0.95$. The lower tolerance limit is equal to 25, the upper tolerance limit is equal to 29 and the coverage is equal to 0.9752. Figure 4 shows the histogram of P_{95}.

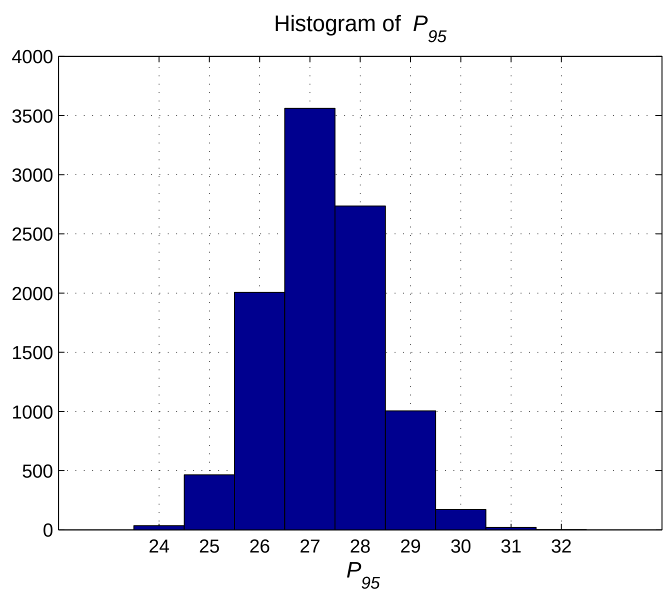

Figure 4: Histogram of P_{95}.

5. Conclusions

Bayesian tolerance intervals for the Poisson distribution were introduced, and from the simulation study it was seen that the coverage rates obtained for one-sided and two-sided intervals were relatively good. For the two-sided and one-sided intervals the coverage rates were most of the time at or above 0.95, except in a few cases. Tolerance intervals could be useful for applications in quality control. An objective Bayesian approach was used.

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