

# Behavior of binned kernel density estimators for non-stationary random fields

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**Abstract** We investigate the asymptotic behavior of binned kernel density estimators for dependent and locally non-stationary random fields *converging* to stationary random fields. We focus on the study of the bias and the asymptotic normality of the estimators. A simulation experiment conducted shows that both the kernel density estimator and the binned kernel density estimator have the same behavior and estimate accurately the true density when the number of fields increases.

## 1 Introduction

In many practical situations, one can be concerned with the statistical study of an unobserved stationary random fields  $(X_{\mathbf{i}}^*)_{\mathbf{i} \in \mathbb{Z}^N}$ ,  $N \in \mathbb{N}$  at the place of which a sequence of random fields  $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^N}$  is observed and both series are linked by an equation of the form

$$X_{\mathbf{i}} = \vartheta(\mathbf{i}) + (1 + \zeta(\mathbf{i}))X_{\mathbf{i}}^*, \quad \mathbf{i} \in \mathbb{Z}^N, \quad (1.1)$$

where  $\vartheta$  and  $\zeta$  are some functions defined on  $\mathbb{Z}^N$ . Denoting by  $f^*$  the density function (with respect to Lebesgue measure) of the stationary distribution of  $(X_{\mathbf{i}}^*)_{\mathbf{i} \in \mathbb{Z}^N}$ , statistical inferences of interest can be testing hypothesis on  $f^*$  or estimating this function. It is clear that such works can only be done through the non-stationary  $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^N}$  defined in (1.1) and studied for instance in [6]. In the present paper, we show that under some conditions, the so-called binned kernel density estimator (BKDE) based on  $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^N}$  is consistent to  $f^*$  and is asymptotically normal. We study the BKDE instead of the classical kernel density estimator (KDE) because of its lower cost computational advantage.

Let  $n_1, \dots, n_N$  be positive integers. Denote by  $\mathbf{n}$  the  $N$ -dimensional vector  $(n_1, \dots, n_N)$  and by  $I_{\mathbf{n}} = \prod_{k=1}^N [1, \dots, n_k]$ , a finite rectangular domain of the integer lattice points in  $N$ -dimensional Euclidian space  $\mathbb{Z}^N$ . Taking in (1.1)  $\vartheta = \zeta = 0$ , the Rosenblatt [7] kernel density estimator  $\hat{f}$  of  $f^*$  is defined by

$$\hat{f}(x) = \frac{1}{\hat{\mathbf{n}}h} \sum_{\mathbf{i} \in I_{\mathbf{n}}} K\left(\frac{x - X_{\mathbf{i}}}{h}\right), \quad x \in \mathbb{R}, \quad (1.2)$$

where  $\hat{\mathbf{n}}$  is the finite product  $n_1 \dots n_N$ ,  $h = h(\mathbf{n})$  is the smoothing parameter and  $K$  is a bounded integrable real-valued function defined on  $\mathbb{R}$ , called kernel.

It is well known that (1.2) has a high computation cost. The most popular way to reduce this cost is to prebin the data, an operation which leads to the BKDE studied for instance in Hall [2], Scott

and Shearter [8], Hall and Wand [1] and Holmström [4]. The BKDE can be seen as approximations of KDE, or as direct estimators of  $f^*$ . They have the general form

$$\tilde{f}(x) = \frac{1}{\hat{\mathbf{n}}h} \sum_{j \in \mathbf{Z}} K\left(\frac{x - a_j}{h}\right) \sum_{\mathbf{i} \in I_{\mathbf{n}}} T\left(\frac{X_{\mathbf{i}} - a_j}{\delta}\right), \quad x \in \mathbb{R}, \quad (1.3)$$

where  $\{a_j\} = \{a_0 + j\delta\}_{j \in \mathbb{Z}}$  is a given grid points with an arbitrary origin  $a_0 \in \mathbb{R}$ ,  $T$  is a kernel with window width  $\delta$ , and  $h$  and  $K$  are as above.

The aim is the study of the behavior of  $\tilde{f}$  for *local non-stationary*  $\alpha$ -mixing random fields. That is, for  $\alpha$ -mixing random fields  $\{X_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^N}$  for which there exists a finite set of neighboring sites  $I_{\mathbf{n}}^* \subset I_{\mathbf{n}}$  such that the sequence  $\{X_{\mathbf{i}}\}_{\mathbf{i} \in I_{\mathbf{n}}^*}$  is possibly non-stationary and the sequence  $\{X_{\mathbf{i}}\}_{\mathbf{i} \in I_{\mathbf{n}} - I_{\mathbf{n}}^*}$  is stationary with a stationary distribution different from at least that of one of the  $X_{\mathbf{i}}$ ,  $\mathbf{i} \in I_{\mathbf{n}}^*$ . For example, for the series satisfying (1.1), we consider the cases where  $\vartheta(\mathbf{i})$  and  $\zeta(\mathbf{i})$  tend to zero as  $\mathbf{i}$  tends to infinity.

Denote by  $[x]$  the integer closest to  $x$  and by  $\lfloor x \rfloor$  the largest integer less than or equal to  $x$ . For  $\delta > 0$  and an arbitrary  $a_0 \in \mathbb{R}$ , define the real-valued function  $a$  by  $a(y) = \delta \lfloor (y - a_0)/\delta \rfloor + a_0$ , for rounding to the nearest value, or  $a(y) = \delta \lfloor (y - a_0)/\delta \rfloor + a_0$ , for rounding down. For a clear presentation, we restrict ourselves to the cases where  $\tilde{f}$  is defined with  $T(y) = I(y \in [0, 1])$  or with  $T(y) = I(y \in [-1/2, 1/2])$ . More precisely, we consider

$$\tilde{f}(x) = \frac{1}{nh} \sum_{\mathbf{i} \in I_{\mathbf{n}}} K\left(\frac{x - X_{\mathbf{i}}}{h} + \frac{\delta}{h} Z_{\mathbf{i}}\right), \quad x \in \mathbb{R}, \quad (1.4)$$

where for  $\mathbf{i} \in I_{\mathbf{n}}$ ,  $Z_{\mathbf{i}} = (X_{\mathbf{i}} - a(X_{\mathbf{i}}))/\delta \in (0, 1)$  for rounding down or  $Z_{\mathbf{i}} \in [-1/2, 1/2]$  for rounding to the nearest value.

In Section 2, we list the notations and the sequence of assumptions considered along the paper. In Section 3, we state our mains results.

## 2 General assumptions

For all  $\mathbf{i} = (i_1, \dots, i_N) \in I_{\mathbf{n}}$ ,  $|\mathbf{i}| = \max_{k=1, \dots, N} |i_k|$ . We use the notation  $\mathbf{n} \rightarrow \infty$  to mean that  $\min_{k=1, \dots, N} n_k \rightarrow \infty$  and  $\max_{j,k=1, \dots, N} (n_j/n_k) < C$  for some generic constant  $C > 0$ . The total variation norm of a real-value function  $\varpi$  is denoted by  $\|\varpi\|_V$ , and if  $\int \varpi^p(x) dx < \infty$ , its  $L_p$ -norm is defined by  $\|\varpi\|_p = (\int \varpi^p(x) dx)^{1/p}$ . For simplicity, we only treat the case where  $I_{\mathbf{n}}^*$  contains only one single site  $\mathbf{i}_0$  of  $I_{\mathbf{n}}$ . We make the following assumptions:

(A1) :

- For all  $\mathbf{i} \in I_{\mathbf{n}}$ ,  $X_{\mathbf{i}}$  has a cumulative distribution function  $F_{\mathbf{i}}$  with density function  $f_{\mathbf{i}}$ , both continuous.
- There exists a strictly stationary random field  $\{X_{\mathbf{i}}^*\}_{\mathbf{i} \in \mathbb{Z}^N}$  with continuous distribution and density functions  $F^*$  and  $f^*$  respectively.
- For  $|\mathbf{j} - \mathbf{i}| > 0$ ,  $(X_{\mathbf{i}}, X_{\mathbf{j}})$  has a continuous distribution function  $F_{\mathbf{i}, \mathbf{j}}$  with a continuous density function  $f_{\mathbf{i}, \mathbf{j}}$ .
- For  $|\mathbf{j} - \mathbf{i}| > 0$ ,  $(X_{\mathbf{i}}^*, X_{\mathbf{j}}^*)$  has a continuous distribution function  $F_{|\mathbf{i}-\mathbf{j}|}^*$  with a continuous density function  $f_{|\mathbf{i}-\mathbf{j}|}^*$ .

- For all  $\mathbf{i} \in I_{\mathbf{n}}$ ,  $(X_{\mathbf{i}}, Z_{\mathbf{i}})$  has a continuous distribution function  $G_{\mathbf{i}}$  with continuous density function  $g_{\mathbf{i}}$ .
- For  $|\mathbf{j} - \mathbf{i}| > 0$ ,  $(X_{\mathbf{i}}, X_{\mathbf{j}}, Z_{\mathbf{i}}, Z_{\mathbf{j}})$  has a continuous distribution function  $G_{\mathbf{i}, \mathbf{j}}$  with continuous density function  $g_{\mathbf{i}, \mathbf{j}}$ .
- In addition

$$\max \left\{ \sup_{x \in \mathbb{R}} |f^{*'}(x)|, \sup_{\mathbf{i} \in \mathbb{Z}^N} \sup_{x \in \mathbb{R}} f_{\mathbf{i}}(x), \sup_{\mathbf{i}, \mathbf{j} \in \mathbb{Z}^N} \left[ \sup_{x, y \in \mathbb{R}} f_{\mathbf{i}, \mathbf{j}}(x, y), \sup_{\substack{x, y \in \mathbb{R} \\ u, v \in \Delta}} g_{\mathbf{i}, \mathbf{j}}(x, y, u, v), \sup_{\substack{x, y \in \mathbb{R} \\ z = x, y}} \left| \frac{\partial f_{|\mathbf{j} - \mathbf{i}|}^*(x, y)}{\partial z} \right| \right] \right\} < \infty,$$

for  $|\mathbf{i} - \mathbf{i}_0| < |\mathbf{j} - \mathbf{i}_0|$  and  $|\mathbf{i} - \mathbf{i}_0| \rightarrow \infty$ ,

$$\|F_{\mathbf{i}, \mathbf{j}} - F_{|\mathbf{j} - \mathbf{i}|}^*\|_{\mathcal{V}} = O(\eta(|\mathbf{i} - \mathbf{i}_0|)) \rightarrow 0 \quad \text{and} \quad \|F_{\mathbf{i}} - F^*\|_{\mathcal{V}} = O(\eta(|\mathbf{i} - \mathbf{i}_0|)) \rightarrow 0.$$

where  $\eta$  is some non-increasing function which satisfies  $\sum_{\mathbf{i} \in I_{\mathbf{n}}} \eta(|\mathbf{i} - \mathbf{i}_0|) < \infty$ , as  $\mathbf{n} \rightarrow \infty$ .

(A2) :

- The nonnegative function  $K$  is bounded, symmetric, absolutely continuous and piecewise differentiable with a bounded derivative, and is such that  $\int K(x)dx = 1$ ,  $\int x^2 K(x)dx$ ,  $\int x K'(x)dx$ ,  $\sup_{x \in \mathbb{R}} |K'(x)|$  and  $\int |K'(x)| dx$  are finite.
- The sequences  $h = h(\hat{\mathbf{n}})$  and  $\delta = \delta(\hat{\mathbf{n}})$  are positive and are such that  $h \rightarrow 0$ ,  $\delta \rightarrow 0$ ,  $\delta/h \rightarrow 0$ ,  $\hat{\mathbf{n}}h \rightarrow \infty$ , as  $\mathbf{n} \rightarrow \infty$ .

(A3) :

- The sequences of random fields  $\{X_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^N}$  and  $\{X_{\mathbf{i}}^*\}_{\mathbf{i} \in \mathbb{Z}^N}$  are  $\alpha$ -mixing with the same mixing rate. That is, for all  $u, v > 0$ ,

$$\max_{U, V \subset \mathbb{Z}^N} \sup_{A \in \mathcal{U}_{U(u)}, B \in \mathcal{V}_{V(v)}} |P(A \cap B) - P(A)P(B)| = \alpha_{u, v}(m) \rightarrow 0 \text{ as } m \rightarrow \infty,$$

where  $\mathcal{U}_{U(u)}$  and  $\mathcal{V}_{V(v)}$  are respectively the  $\sigma$ -algebras spanned by  $(X_{\mathbf{i}}, \mathbf{i} \in U, \text{Card}(U) \leq u)$  and  $(X_{\mathbf{j}}, \mathbf{j} \in V, \text{Card}(V) \leq v)$  with  $\inf_{\mathbf{i} \in U(u), \mathbf{j} \in V(v)} |\mathbf{i} - \mathbf{j}| \geq m$ , and  $\alpha_{u, v}(m)$  is an increasing function of  $u$  and  $v$ , and a decreasing function of  $m$ . Here, we take  $\alpha_{u, v}(n) = \zeta_{u, v} \alpha(n) \rightarrow 0$  as  $n \rightarrow \infty$ , for all  $u$  and  $v > 0$ , where  $\zeta_{u, v}$  is an increasing function of  $u$  and  $v$ , and  $\alpha(n)$  is a decreasing function of  $n$ .

- There exist  $\nu > 0$  and  $c \geq 3$ , such that for all  $u, v \in \mathbb{N}^*$ ,  $u + v \leq c$ ,  $u, v \geq 2$ ,  $\sum_{r \geq 1} (r + 1)^{N(c-u+1)-1} [\alpha(r)]^{1/\lambda} < \infty$  with  $\lambda = (c + \nu)/\nu$ , and for  $h = h(\hat{\mathbf{n}})$  there exists an increasing sequence  $m = m(\hat{\mathbf{n}})$  satisfying  $hm^N \rightarrow 0$ , and  $h^{-1/\lambda} \sum_{k=m}^{\infty} k^{N-1} \alpha(k)^{1/\lambda} \rightarrow 0$  as  $\mathbf{n} \rightarrow \infty$ .

### 3 Main results

Our main results are on the study of the bias of  $\tilde{f}$  with respect to the stationary density  $f^*$ . The proofs of our results are based on the same techniques as those of [5] and [3].

### 3.1 Bias and other convergence results

We start by the study of the bias. Our results on this are stated through the following propositions and theorem which can be established by classical techniques.

**Proposition 1** *Under the assumptions (A1)-(A2), the binned kernel estimator  $\tilde{f}$  is an asymptotically unbiased estimator of  $f^*$ . Moreover,*

$$E[\tilde{f}(x)] - f^*(x) = E[\hat{f}^*(x) - f^*(x)] + O\left(\frac{\delta}{h} + \frac{1}{\hat{\mathbf{n}}h}\right) \quad (3.1)$$

where  $E[\hat{f}^*(x) - f^*(x)]$  stands for the bias of the Rosenblatt estimator of  $f^*$  based on  $(X_{\mathbf{i}}^*)_{\mathbf{i} \in \mathbb{Z}^N}$ .

**Proposition 2** *Under the assumptions (A1) and (A2)*

$$MISE(\tilde{f}(x)) - MISE(\hat{f}^*(x)) = O\left(\frac{1}{\hat{\mathbf{n}}} + \frac{\delta}{h} + \delta \int_{\Delta} u du + \frac{1}{\hat{\mathbf{n}}h}\right).$$

For  $m = o(\hat{\mathbf{n}}^{1/N})$ , if

$$\frac{1}{\hat{\mathbf{n}}h} \sum_{k=m}^{\infty} k^{N-1} \alpha(k)^{1/\lambda} \longrightarrow 0 \quad \text{as } \mathbf{n} \rightarrow \infty,$$

then  $\int E[\tilde{f}(x) - f^*(x)]^2 dx$  tends to 0 as  $\mathbf{n}$  tends to infinity.

**Proposition 3** *Under the assumptions (A1)-(A3), for  $m = o(\hat{\mathbf{n}}^{1/N})$  the mean square quadratic difference between  $\tilde{f}$  and  $\hat{f}$  is given by :*

$$E\left\{\left[\tilde{f}(x) - \hat{f}(x)\right]^2\right\} = O\left(\frac{\delta^2}{h^2}\right).$$

**Theorem 1** *Under the assumptions (A1) and (A2), for  $m = o(\hat{\mathbf{n}}^{1/N})$ , if*

$$\frac{1}{\hat{\mathbf{n}}h} h^{-1/\lambda} \sum_{k=m}^{\infty} k^{N-1} \alpha(k)^{1/\lambda} \longrightarrow 0 \quad \text{as } \mathbf{n} \rightarrow \infty,$$

then the binned kernel estimator  $\tilde{f}$  converges in mean square to  $f^*$ .

Theorem 1 is an immediate consequence of the following triangle inequality

$$\sqrt{E[\tilde{f}(x) - f^*(x)]^2} \leq \sqrt{E[\tilde{f}(x) - \hat{f}(x)]^2} + \sqrt{E[\hat{f}(x) - f^*(x)]^2}, \quad (3.2)$$

Proposition 3 and the following lemma.

**Lemma 1** Under the assumptions (A1) and (A2), if there exists  $m = o(\widehat{\mathbf{n}}^{1/N})$  such that

$$\frac{1}{\widehat{\mathbf{n}}h} h^{-1/\lambda} \sum_{k=m}^{\infty} k^{N-1} \alpha(k)^{1/\lambda} \rightarrow 0 \text{ as } \mathbf{n} \rightarrow \infty,$$

then the Rosenblatt estimator  $\widehat{f}$  converges to  $f^*$  in mean square. Moreover, if assumption (A3) is satisfied, then

$$\begin{aligned} E \left[ \widehat{f}(x) - f^*(x) \right]^2 &= \frac{1}{\widehat{\mathbf{n}}h} \|K\|_2^2 f^*(x) + o(\widehat{\mathbf{n}}^{-1}) \\ &+ O \left( \frac{m^N}{\widehat{\mathbf{n}}} + \frac{1}{\widehat{\mathbf{n}}h} h^{-1/\lambda} \sum_{k=m}^{\infty} k^{N-1} \alpha_{1,1}(k)^{1/\lambda} + \frac{1}{\widehat{\mathbf{n}}^2 h^2} \right). \end{aligned} \quad (3.3)$$

Proposition 3 shows that the first term in the right-hand side of (3.2) tends to zero, and Lemma 1 shows that the second term also tends to zero. This establishes Theorem 1.

### 3.2 Central Limit Theorem

Denote by  $A_{\mathbf{i}} = K[(x - X_{\mathbf{i}})/h]$  and  $H_{\mathbf{i}} = A_{\mathbf{i}} - E(A_{\mathbf{i}})$ ,  $\mathbf{i} \in I_{\mathbf{n}}$ . One has the following lemma:

**Lemma 2** Let  $\widetilde{I}_{\mathbf{n}} \subset I_{\mathbf{n}}$  and  $\#(\widetilde{I}_{\mathbf{n}})$  the number of elements in  $\widetilde{I}_{\mathbf{n}}$ . Under the assumptions (A1)-(A3), for some  $p$  such that  $1/\lambda = 1 - 2/p$ ,

$$\begin{aligned} E \left[ \left( \sum_{\mathbf{i} \in \widetilde{I}_{\mathbf{n}}} H_{\mathbf{i}} \right)^2 \right] &= O \left( \left( C + \#(\widetilde{I}_{\mathbf{n}}) \frac{\delta^2}{h} \right)^2 + Ch \left( C + \#(\widetilde{I}_{\mathbf{n}}) \frac{\delta^2}{h} \right) + \#(\widetilde{I}_{\mathbf{n}}) \left( \frac{\delta^2}{h} + \delta + \frac{\delta^2}{h} \right) \right) \\ &+ \#(\widetilde{I}_{\mathbf{n}}) [h f^*(x) \|K\|_2^2 + O(h^2)] \\ &+ O \left( \#(\widetilde{I}_{\mathbf{n}}) m^N h^2 + h m^N \right) + O \left( \#(\widetilde{I}_{\mathbf{n}}) h^{2/p} \sum_{k=m}^{\infty} k^{N-1} \alpha_{1,1}(k)^{1/\lambda} \right). \end{aligned} \quad (3.4)$$

$$E \left( \left| \sum_{\mathbf{i} \in \widetilde{I}_{\mathbf{n}}} H_{\mathbf{i}} \right|^3 \right) = O \left( \#(\widetilde{I}_{\mathbf{n}})^{3/2} h \right). \quad (3.5)$$

Let  $q_k = n_k/(\ell + m)$  with  $m = o(\ell)$  and  $\ell = \widehat{\mathbf{n}}^{(1-\beta)/N}$  for some  $\beta \in (0, 1)$ , such that  $m$  and  $q_k$  are integers. One has  $\widehat{\mathbf{q}} = q_1 \dots q_N = \widehat{\mathbf{n}}/(\ell + m)^N$  cubes in  $I_{\mathbf{n}}$ . Let  $Q_{\mathbf{n}}$  be the subset of  $\mathbb{Z}^N$  indexing these  $\widehat{\mathbf{q}}$  cubes. Each cube of length  $(\ell + m)^N$  contains  $2^N$  elements with length varying from  $\ell^N$  to  $m^N$ . This blocking separates the cubes of length  $\ell^N$  by “envelopes” of length  $(\ell + m)^N - \ell^N = O(Nm\ell^{N-1})$ . Define

$$S_{\mathbf{n}} = \sqrt{\widehat{\mathbf{n}}h} \left[ \widetilde{f}(x) - E\widetilde{f}(x) \right] \quad \text{and} \quad \varphi_{\mathbf{n}}(t) = E[\exp(itS_{\mathbf{n}})].$$

Then, recalling that for  $\mathbf{i} \in I_{\mathbf{n}}$ ,  $A_{\mathbf{i}} = K[(x - a(X_{\mathbf{i}}))/h + \delta Z_{\mathbf{i}}/h]$  and  $H_{\mathbf{i}} = A_{\mathbf{i}} - EA_{\mathbf{i}}$ , one can write

$$S_{\mathbf{n}} = \frac{1}{\sqrt{\widehat{\mathbf{n}}h}} \sum_{\mathbf{i} \in I_{\mathbf{n}}} H_{\mathbf{i}} = \sum_{\mathbf{j} \in Q_{\mathbf{n}}} U_{\mathbf{j}} + \sum_{\mathbf{j} \in Q_{\mathbf{n}}} V_{\mathbf{j}},$$

where  $U_{\mathbf{j}}$  is a sum of  $\ell^N$  terms and  $V_{\mathbf{j}}$  is a sum of  $(\ell + m)^N - \ell^N$  terms. Using this decomposition, one obtains :

**Lemma 3** Under the assumptions (A1)-(A3), as  $\mathbf{n} \rightarrow \infty$ , one has the inequality

$$\left| \varphi_{\mathbf{n}}(t) - \prod_{\mathbf{j} \in Q_{\mathbf{n}}} E[\exp(it U_{\mathbf{j}})] \right| \leq C\sqrt{m\ell^{-1}} + CN\hat{\mathbf{q}}\zeta_{\ell^N, \ell^N} \left[ \alpha(m) + \sum_{i=1}^{\max_k(q_k)} i^{N-1} \alpha(i(m+\ell)) \right]. \quad (3.6)$$

If in addition,  $\zeta_{u,v} \leq \min\{u, v\}$  the above inequality becomes

$$\left| \varphi_{\mathbf{n}}(t) - \prod_{\mathbf{j} \in Q_{\mathbf{n}}} E[\exp(it U_{\mathbf{j}})] \right| \leq C\sqrt{m\ell^{-1}} + C\hat{\mathbf{n}}\alpha(m). \quad (3.7)$$

**Theorem 2** Assume that (A1)-(A3) hold. If

$$\hat{\mathbf{n}}^\beta \zeta_{\ell^N, \ell^N} \left[ \alpha(m) + \sum_{i=1}^{\infty} i^{N-1} \alpha(i(m+\ell)) \right] \rightarrow 0, \quad \mathbf{n} \rightarrow \infty, \quad (3.8)$$

then for  $h = Cn^{\beta_0}$ ,  $0 < \beta_0 < \beta$ ,  $S_{\mathbf{n}}$  converges in distribution to a zero-mean Gaussian random variable with variance  $\sigma^2(x) = f^*(x) \|K\|_2^2$ .

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