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Tail dependence convergence rate of a skew-t and of a skew normal distribution

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(Lower) Tail dependence

- The coefficient of lower tail dependence of a random vector $\mathbf{X} = (X_1, X_2)^\top$, is defined as

$$\begin{aligned} \lambda_L &= \lim_{u \to 0^+} \lambda_L(u), \\ \text{where} \quad \lambda_L(u) &= P(X_1 \leq F_1^{-1}(u) | X_2 \leq F_2^{-1}(u)), \end{aligned}$$

  where F_1^{-1} and F_2^{-1} denote the marginal inverse distribution functions (assuming they are well defined).

- $\mathbf{X}$ is said to have asymptotic lower tail dependence if λ_L exists and is positive. If $\lambda_L = 0$ then $\mathbf{X}$ is said to be asymptotically independent in the lower tail.

- These quantities provide insight on the tendency for the distribution to generate joint extreme event since they measure the strength of dependence (or association) in the tails of a bivariate distribution.

Skew normal and skew-t distributions

- A random vector $\mathbf{Z}$ is said to have a bivariate skew normal distribution, denoted as $\mathbf{Z} \sim SN_2(\boldsymbol{\theta}, R)$, if it has density

$$f(\mathbf{z}) = 2\phi_2(\mathrm{z}, R)\Phi(\boldsymbol{\theta}^\top \mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^2 \tag{1}$$

  where $\phi_2(\cdot, R)$ is the bivariate normal density with mean $\mathbf{0}$ and correlation matrix $R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, $\Phi(\cdot)$ is the cdf of $N(0, 1)$ and $\boldsymbol{\theta} = (\theta_1, \theta_2)^\top$ is a vector that controls the asymmetry of the distribution.

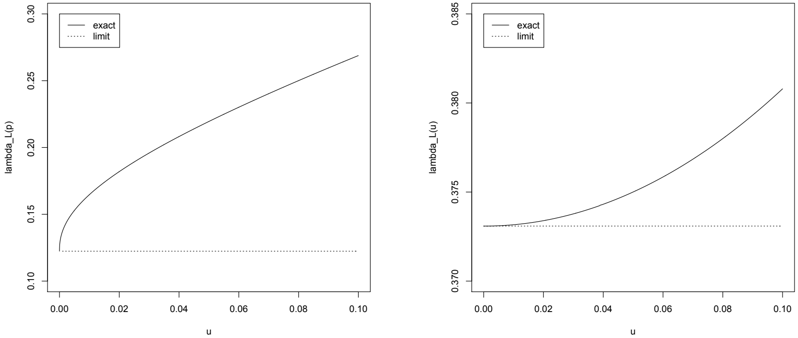
- We define the bivariate skew-t as that distribution resulting from variance-mixing of the bivariate skew normal, $\mathbf{Z} \sim SN_2(\boldsymbol{\theta}, R)$, inversely with a gamma random variable $V \sim \Gamma(\frac{\eta}{2}, \frac{\eta}{2})$, with $\eta > 0$:

$$\mathbf{X} = V^{-\frac{1}{2}}\mathbf{Z}, \tag{2}$$

  where $\mathbf{Z}$ is independently distributed of V.

$\lambda_L(u)$ and $\lambda_L = \lim_{u \to 0^+} \lambda_L(u)$

- The bivariate skew-t always satisfies $\lambda_L > 0$. See Fung and Seneta (2010) for instance.



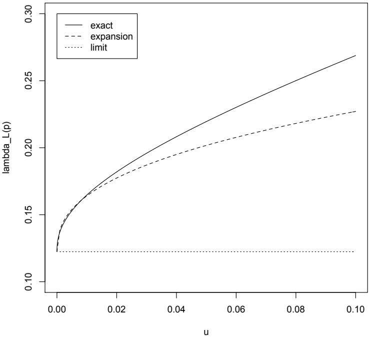
(a) (b)

Figure 1: Exact $\lambda_L(u)$ against u for the skew-t distribution with $\rho = 0.3$, $\boldsymbol{\theta}^\top = (0.1, 0.3)$ and $\lambda_L = \lim_{u \to 0^+} \lambda_L(u)$ for (a) $\eta = 5$; (b) $\eta = 1$.

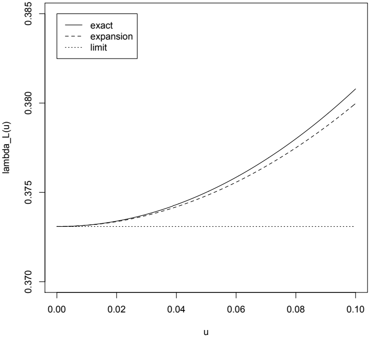
Rate of Convergence

- We have considered the rate of convergence to 0 as $u \to 0^+$ of $|\lambda_L(u) - \lambda_L|$ i.e.

$$|\lambda_L(u) - \lambda_L| = u^{\kappa} L(u).$$



(a)



(b)

Figure 2: Exact and approximation with correction term for $\lambda_L(u)$ against u for the skew t distribution with $\rho = 0.3$, $\boldsymbol{\theta}^\top = (0.1, 0.3)$ for (a) $\eta = 5$; (b) $\eta = 1$.

- There are more discussion and some results on the skew normal distribution on the poster.
- Thank you for listening!