Wild Bootstrap Tests for Autocorrelation in Vector Autoregressive Models

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Abstract

Tests for error autocorrelation (AC) are derived under the assumption of independent and identically distributed (IID) errors. The tests are not asymptotically valid if the errors are conditionally heteroskedastic. In this paper we propose wild bootstrap (WB) Lagrange multiplier tests for error AC in vector autoregressive (VAR) models. We show that the WB tests are asymptotically valid under conditional heteroskedasticity of unknown form. WB tests based on a version of the heteroskedasticity-consistent covariance matrix estimator are found to have the smallest error in rejection probability under the null and high power under the alternative. We apply the tests to VAR models for credit default swap (CDS) prices and Euribor interest rates. An important result that we find is that the WB tests lead to parsimonious models while the asymptotic tests suggest that a long lag length is required to get white noise residuals.

Keywords: autocorrelation; conditional heteroskedasticity; Lagrange multiplier test; vector autoregressive model; wild bootstrap.

1. Introduction

There is a lot of empirical evidence against the assumption of independent and identically distributed (IID) errors in time series models of economic and financial time series (Gonçalves and Kilian, 2004; Hafner and Herwartz, 2009; Ahlgren and Catani, 2014).

Tests for error autocorrelation (AC) are derived under the assumption of IID errors and are not asymptotically valid if the errors are (conditionally) heteroskedastic. A solution to the problem with (conditional) heteroskedasticity is to employ heteroskedasticity-consistent covariance matrix estimators (HCCMEs) (Eickert, 1963; White, 1980). The HCCME leads to tests which are asymptotically valid under (conditional) heteroskedasticity.

Wild bootstrap (WB) procedures for stationary autoregressions with conditional heteroskedasticity are proposed by Gonçalves and Kilian (2004). Hafner and Herwartz (2009) propose a fixed-design WB procedure to test parameter restrictions in vector autoregressive (VAR) models that is robust under conditional heteroskedasticity of unknown form. Brüggemann et al. (2014) extend the WB procedures of Gonçalves and Kilian to VAR models.

In this paper we propose WB Lagrange multiplier (LM) tests for error AC in VAR models. We show that the WB tests are asymptotically valid under conditional heteroskedasticity of unknown form. We consider a number of possible forms of the HCCME, and investigate the finite-sample properties of the tests by Monte Carlo simulations. The simulation experiments provide evidence that the asymptotic tests for error AC are severely oversized, whereas HCCME-based tests are undersized and have low power in the presence of strong persistence in volatility in the form of autoregressive conditional heteroskedasticity (ARCH) errors. The WB tests perform well both when the errors are IID and conditionally heteroskedastic. WB tests based on a version of the heteroskedasticity-consistent covariance matrix estimator are found to have the smallest error in rejection probability under the null and high power under the alternative. In two empirical applications, we consider asymptotic and WB tests for error AC in VAR models for credit default swap (CDS) prices and

Euribor interest rates.

2. Tests for Error Autocorrelation

The K-dimensional vector of $I(0)$ time series variables $\mathbf{y}_t$ is assumed to be generated by a vector autoregressive (VAR) model of order p:

$$\mathbf{y}_t = \mathbf{A}_1\mathbf{y}_{t-1} + \cdots + \mathbf{A}_p\mathbf{y}_{t-p} + \mathbf{u}_t, \qquad t = 1, \ldots, T. \tag{1}$$

Here $\mathbf{A}_1, \ldots, \mathbf{A}_p$ are $(K \times K)$ parameter matrices. The error process $\{\mathbf{u}_t\}$ is assumed to be IID with mean zero, and nonsingular and positive definite covariance matrix $\mathbf{\Sigma_u}$. The assumption of IID errors is used to derive the basic form of the LM statistic and its limiting distribution under the null. The alternative is a VAR(h) model for the errors:

$$\mathbf{u}_t = \mathbf{D}_1\mathbf{u}_{t-1} + \cdots + \mathbf{D}_h\mathbf{u}_{t-h} + \mathbf{e}_t. \tag{2}$$

The hypothesis being tested is $H_0 : \mathbf{D}_1 = \cdots = \mathbf{D}_h = \mathbf{0}$ against $H_1 : \mathbf{D}_j \neq \mathbf{0}$ for at least one $j \in \{1, \ldots, h\}$. The LM statistic is computed from an auxiliary model

$$\begin{aligned} \widehat{\mathbf{u}}_t &= \mathbf{A}_1\mathbf{y}_{t-1} + \cdots + \mathbf{A}_p\mathbf{y}_{t-p} + \mathbf{D}_1\widehat{\mathbf{u}}_{t-1} + \cdots + \mathbf{D}_h\widehat{\mathbf{u}}_{t-h} + \mathbf{e}_t \\ &= (\mathbf{Z}_t' \otimes \mathbf{I}_K)\phi + (\widehat{\mathbf{U}}_t' \otimes \mathbf{I}_K)\psi + \mathbf{e}_t, \end{aligned} \tag{3}$$

where $\mathbf{Z}_t' = (\mathbf{y}_{t-1}', \ldots, \mathbf{y}_{t-p}')$, $\phi = \text{vec}(\mathbf{A}_1, \ldots, \mathbf{A}_p)'$, $\widehat{\mathbf{U}}_t' = (\widehat{\mathbf{u}}_{t-1}', \ldots, \widehat{\mathbf{u}}_{t-h}')$ and $\psi = \text{vec}(\mathbf{D}_1, \ldots, \mathbf{D}_h)'$. The symbol $\otimes$ denotes the Kronecker product and the symbol vec denotes the column vectorisation operator.

The LM statistic is given by (see e.g. Godfrey, 1991)

$$Q_{LM}(h) = T\widehat{\psi}'(\widehat{\mathbf{\Sigma}}^{\psi\psi})^{-1}\widehat{\psi}, \tag{4}$$

where $\widehat{\psi}$ is the least squares (LS) estimate of ψ and $\widehat{\mathbf{\Sigma}}^{\psi\psi}$ is the block of

$$\left(T^{-1} \sum_{t=1}^{T} \begin{bmatrix} \mathbf{Z}_t \otimes \mathbf{I}_K \\ \widehat{\mathbf{U}}_t \otimes \mathbf{I}_K \end{bmatrix} \widehat{\mathbf{\Sigma}}_{\mathbf{u}}^{-1} \begin{bmatrix} \mathbf{Z}_t' \otimes \mathbf{I}_K & \widehat{\mathbf{U}}_t' \otimes \mathbf{I}_K \end{bmatrix} \right)^{-1}$$

corresponding to ψ. Here $\widehat{\mathbf{\Sigma}}_{\mathbf{u}} = T^{-1}\sum_{t=1}^{T} \widehat{\mathbf{u}}_t\widehat{\mathbf{u}}_t'$ is the estimator of the error covariance matrix from the original VAR model. The Q_{LM} statistic is asymptotically distributed as χ^2 with hK^2 degrees of freedom under the null hypothesis. The test reduces to the single equation LM test when $K = 1$.

Brüggemann et at. (2006) show that LM tests for error AC are valid in integrated and cointegrated VAR models with IID errors. The LM statistic has the same asymptotic distribution as in stationary VAR models.

The standard LS-based LM statistic in (4) is not robust against conditional heteroskedasticity. We employ the multivariate HCCME as in Hafner and Herwartz (2009). The multivariate HCCME for the auxiliary model in (3) is given by

$$\mathbf{V}_T^{-1}\mathbf{W}_T\mathbf{V}_T^{-1} = (\mathbf{\Gamma}_T \otimes \mathbf{I}_K)^{-1}\mathbf{W}_T(\mathbf{\Gamma}_T \otimes \mathbf{I}_K)^{-1}, \tag{5}$$

where

$$\mathbf{V}_T = \mathbf{\Gamma}_T \otimes \mathbf{I}_K, \tag{6}$$

$$\mathbf{\Gamma}_T = \frac{1}{T}\sum_{t=1}^{T} \begin{pmatrix} \widehat{\mathbf{U}}_t \\ \mathbf{Z}_t \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{U}}_t' & \mathbf{Z}_t' \end{pmatrix}, \tag{7}$$

$$\mathbf{W}_T = \frac{1}{T}\sum_{t=1}^{T} \begin{pmatrix} \widehat{\mathbf{U}}_t \\ \mathbf{Z}_t \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{U}}_t' & \mathbf{Z}_t' \end{pmatrix} \otimes (\widehat{\mathbf{u}}_t\widehat{\mathbf{u}}_t'). \tag{8}$$

MacKinnon and White (1985) refer to the basic version (5) of the HCCME as HC_0 and consider three modified versions of HC_0, which they denote by HC_1, HC_2 and HC_3. Rescaling $\widehat{\mathbf{u}}_t$ by $\sqrt{T/(T-Kp)}$ leads to the multivariate analogue of the form HC_1 of the HCCME. Replacing $\widehat{\mathbf{u}}_t$ by $\widehat{\mathbf{u}}_t/(1-h_t)^{1/2}$, where $h_t = \mathbf{Z}_t(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}_t'$ and $\mathbf{Z} = (\mathbf{Z}_0, \ldots, \mathbf{Z}_{T-1})$, we obtain the multivariate analogue of the form HC_2 of the HCCME. The final form of the HCCME that we consider is HC_3, based on arguments from the jackknife, in which $\widehat{\mathbf{u}}_t$ is replaced by $\widehat{\mathbf{u}}_t/(1-h_t)$.

The HCCME-based LM statistics for error AC are obtained from (4) by replacing $\widehat{\boldsymbol{\Sigma}}^{\psi\psi}$ by the block of $\mathbf{V}_T^{-1}\mathbf{W}_T\mathbf{V}_T^{-1} = (\boldsymbol{\Gamma}_T \otimes \mathbf{I}_K)^{-1}\mathbf{W}_T(\boldsymbol{\Gamma}_T \otimes \mathbf{I}_K)^{-1}$ in (5) corresponding to ψ. We denote the HCCME-based LM tests by Q_{LM,HC_0}, Q_{LM,HC_1}, Q_{LM,HC_2} and Q_{LM,HC_3}, respectively.

3. Wild Bootstrap Tests for Error Autocorrelation

We use the recursive-design and fixed-design WB procedures for autoregressions of Gonçalves and Kilian (2004).

For the asymptotic validity of the WB procedures in VAR models, the following assumption from Brüggemann et al. (2014) is made. The assumption is the multivariate analogue of Assumption A in Gonçalves and Kilian.

Assumption 1.

(i) $\mathrm{E}(\mathbf{u}_t|\mathcal{F}_{t-1}) = \mathbf{0}$ almost surely, where $\mathcal{F}_{t-1} = \sigma(\mathbf{u}_{t-1}, \mathbf{u}_{t-2}, \ldots)$ is the σ-field generated by $\{\mathbf{u}_{t-1}, \mathbf{u}_{t-2}, \ldots\}$.

(ii) $\mathrm{E}(\mathbf{u}_t\mathbf{u}_t') = \boldsymbol{\Sigma}_\mathbf{u}$ exists and is positive definite.

(iii) $\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} \mathrm{E}(\mathbf{u}_t\mathbf{u}_t'|\mathcal{F}_{t-1}) = \boldsymbol{\Sigma}_\mathbf{u}$ in probability.

(iv) Define the matrices

$$\tau_{0,a,b,c} = \mathrm{E}(\mathrm{vec}(\mathbf{u}_t\mathbf{u}_{t-a}')\mathrm{vec}(\mathbf{u}_{t-b}\mathbf{u}_{t-c}')') \quad (9)$$

and assume that the elements of $\tau_{0,r,0,s}$ are uniformly bounded for all $r, s \geq 1$. The matrix $\mathbf{L}_K\tau_{0,r,0,r}\mathbf{L}_K'$ for all $r \geq 1$ is positive definite, where $\mathbf{L}_K$ is a $(\frac{1}{2}K(K+1) \times K^2)$ elimination matrix.

(v) $\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} \mathrm{E}(\mathrm{vec}(\mathbf{u}_t\mathbf{u}_{t-r}')\mathrm{vec}(\mathbf{u}_t\mathbf{u}_{t-s}')'|\mathcal{F}_{t-1}) = \tau_{0,r,0,s}$ in probability for all $r, s \geq 1$.

(vi) $\mathrm{E}|\mathbf{u}_t|^{4r}$ is uniformly bounded for some $r \geq 2$.

Assumptions (i) and (ii) replace the IID assumption on the errors $\{\mathbf{u}_t\}$ by the martingale difference (MD) sequence assumption. Assumption (iii) requires convergence of conditional moments. Assumptions (iv) and (v) restrict the fourth-order cumulants of $\mathbf{u}_t$. Assumption (vi) requires the existence of at least 8th moments of the MD sequence $\{\mathbf{u}_t\}$.

The recursive-design WB LM test for error AC is detailed in Algorithm 1.

Algorithm 1 (Recursive-design WB LM test for error AC).

1. Compute the LM statistic Q_{LM} from the data. Obtain the parameter estimates and the residuals $\widehat{\mathbf{u}}_t$ from the VAR model.

2. Draw w_t, $t = 1, ..., T$, independently from a Rademacher distribution and construct the WB errors as $\mathbf{u}_t^* = w_t\widehat{\mathbf{u}}_t$.

3. Generate a bootstrap sample $\{\mathbf{y}_t^*\}$ recursively from

$$\mathbf{y}_t^* = \widehat{\mathbf{A}}_1 \mathbf{y}_{t-1}^* + \cdots + \widehat{\mathbf{A}}_p \mathbf{y}_{t-p}^* + \mathbf{u}_t^*, \tag{10}$$

initialised at $\mathbf{y}_t^* = \mathbf{y}_t$, $t = 1, \ldots, p$.

4. Compute the bootstrap LM statistic Q_{LM}^{*r} from the bootstrap sample $\{\mathbf{y}_t^*\}$. Define the bootstrap p-value as $p^{*r} = 1 - G^*(Q_{LM}^{*r})$, where $G^*(\cdot)$ denotes the conditional (on the original data) cumulative distribution function (CDF) of Q_{LM}^{*r}.

5. The bootstrap test of $H_0 : \mathbf{D}_1 = \cdots = \mathbf{D}_h = \mathbf{0}$ against $H_1 : \mathbf{D}_j \neq \mathbf{0}$ for at least one $j \in \{1, \ldots, h\}$ at the level α rejects H_0 if $p^{*r} \leq \alpha$.

Proposition 1 states the asymptotic validity of the Q_{LM}^{*r} test.

Proposition 1 Under Assumption 1 and under H_0, as $T \to \infty$,

$$\sup_{0<c<\infty} \left| \mathrm{P}^*(Q_{LM}^{*r} \leq c) - \mathrm{P}(Q_{LM} \leq c) \right| \to 0,$$

where P^* denotes the bootstrap probability measure.

The fixed-design WB LM test for error AC is detailed in Algorithm 2.

Algorithm 2 (Fixed-design WB LM test for error AC).

1.–2. Same as Algorithm 1.

3. Generate a bootstrap sample $\{\mathbf{y}_t^*\}$ from

$$\mathbf{y}_t^* = \widehat{\mathbf{A}}_1 \mathbf{y}_{t-1} + \cdots + \widehat{\mathbf{A}}_p \mathbf{y}_{t-p} + \mathbf{u}_t^*, \tag{11}$$

initialised at $\mathbf{y}_t^* = \mathbf{y}_t$, $t = 1, \ldots, p$.

4.–5. Same as Algorithm 1.

Proposition 2 states the asymptotic validity of the Q_{LM}^{*f} test.

Proposition 2 Under Assumption 1 and under H_0, as $T \to \infty$,

$$\sup_{0<c<\infty} \left| \mathrm{P}^*(Q_{LM}^{*f} \leq c) - \mathrm{P}(Q_{LM} \leq c) \right| \to 0,$$

where P^* denotes the bootstrap probability measure.

4. Simulations

Table 1 shows the simulated size of the asymptotic Q_{LM} test, HCCME-based Q_{LM,HC_3} test, recursive-design WB Q_{LM}^{*r} test, HCCME-based recursive-design WB Q_{LM,HC_3}^{*r} test, fixed–design WB Q_{LM}^{*f} test and HCCME-based fixed-design WB Q_{LM,HC_3}^{*f} test for $K = 2$, $T = 100$, 200, 500, 1000, $h = 1$, 4 and 12, and constant conditional correlation generalised autoregressive conditional heteroskedasticity (CCC-GARCH$(1, 1)$) errors with very high persistence in volatility (ARCH parameters 0.08, GARCH parameters 0.9 and conditional correlation coefficient 0). The asymptotic Q_{LM} test is oversized and the size distortions increase with the series length. The HCCME-based $Q_{LM,HC}$ tests are severely undersized in small samples and when the order of AC tested is large. The recursive-design WB Q_{LM}^{*r} test and fixed-design WB Q_{LM}^{*f} test both have

Table 1: Simulated size of asymptotic and WB tests for error AC in stationary VAR(2) model. The dimensions are $K = 2$. The nominal significance level is 5%.

T	100	200	500	1000	100	200	500	1000	100	200	500	1000
		$h=1$				$h=4$				$h=12$		
Q_{LM}	0.076	0.079	0.094	0.099	0.093	0.104	0.124	0.147	0.081	0.109	0.165	0.209
Q_{LM,HC_3}	0.026	0.036	0.047	0.045	0.006	0.018	0.031	0.039	0.000	0.005	0.017	0.026
Q^{*r}_{LM}	0.047	0.048	0.049	0.051	0.044	0.048	0.053	0.051	0.049	0.048	0.049	0.049
Q^{*r}_{LM,HC_3}	0.059	0.052	0.050	0.049	0.055	0.055	0.049	0.053	0.049	0.051	0.049	0.047
Q^{*f}_{LM}	0.054	0.051	0.053	0.051	0.052	0.049	0.050	0.046	0.051	0.052	0.044	0.045
Q^{*f}_{LM,HC_3}	0.046	0.048	0.055	0.049	0.036	0.040	0.043	0.046	0.039	0.037	0.037	0.038

good size properties. The HCCME-based WB tests have better size properties than the WB tests without the HCCME. The HCCME-based recursive-design WB $Q^{*r}_{LM,HC}$ tests have better size properties than the HCCME-based fixed-design $Q^{*f}_{LM,HC}$ tests.

Figure 1 presents the power functions for $h = 1$, 4 and 12 when $T = 100$. The power functions of the tests show that Q^{*r}_{LM} has low power for $h = 1$ and no power at all for $h = 4$ and 12, while Q^{*f}_{LM} has low power. The HCCME-based WB tests are more powerful than the tests without the HCCME.

In summary, the asymptotic Q_{LM} test is not valid under conditional heteroskedasticity. The $Q_{LM,HC}$ tests have low power when the number of observations is small, the dimensions are large and the order of AC tested is large. The WB Q^{*}_{LM} tests outperform the HCCME-based tests in terms of power. The fixed-design WB Q^{*f}_{LM} test is more powerful than the recursive-design WB Q^{*r}_{LM} test. The Q^{*f}_{LM} test has the best performance among all tests when the sample size is small, the dimensions are large and the order of AC tested is large. It has good size and power properties in all cases. The differences in power between the WB tests diminish when the HCCME is used. If the number of observations is large relative to the dimensions and the order of autocorrelation tested, the recursive-design WB Q^{*r}_{LM,HC_3} test has the best performance of all tests in terms of both size and power. However, the Q^{*r}_{LM} test without the HCCME has poor power properties in small samples.

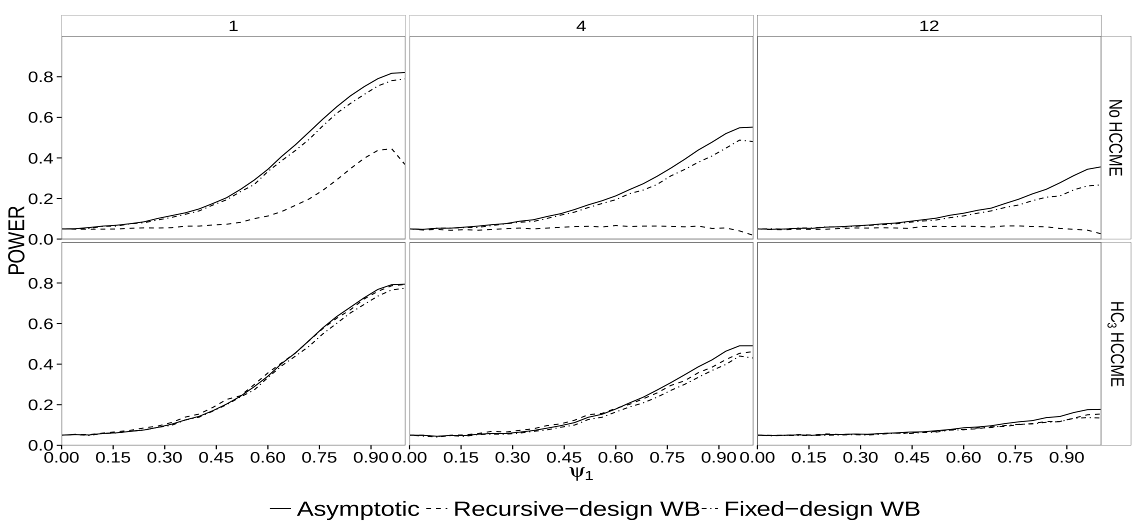

Figure 1: Simulated power functions of the asymptotic and WB tests for error AC in stationary VAR(2) model with $K = 2$ and $T = 100$

5. Empirical Examples

We illustrate the use of the WB LM tests for error AC with real data in empirical applications to credit default swap (CDS) prices and Euribor interest rates. In the empirical example of CDS prices where the sample size is large, we find that the asymptotic test may falsely reject the null hypothesis of no error AC if the errors are conditionally heteroskedastic, whereas based on WB tests the VAR models selected by information criteria provide good descriptions of the data. In the empirical example of Euribor interest rates the sample size is small and the dimensions are large. It is found that the fixed-design WB test without the HCCME is the test for error AC which is most useful.

6. Conclusions

We propose WB LM tests for error AC in VAR models with conditionally heteroskedastic errors. Simulation experiments indicate that asymptotic tests for error AC are severely oversized when the errors are conditionally heteroskedastic. WB tests for error AC perform well whether the errors are IID or conditionally heteroskedastic. The fixed-design WB test without the HCCME has the best performance among all tests when the sample size is small, the dimensions are large and the order of AC tested is large. We therefore recommend that this version of the WB tests should be used in small samples. In large samples the HCCME-based recursive-design WB tests have better size and power properties than other versions of the tests. In large samples we recommend the use of the HCCME-based recursive-design WB test.

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