Verification of absolute nonsingularity of 3-tensors through positivity of a single variate characteristic function

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Abstract

Tensor (multi-way array) data analysis is now a rapidly developing field in statistics, which can analyze more complex data than matrices. Tensor rank is an index of complexity of a datum, and rank determination of tensors is one of fundamental problems in tensor data analysis. Maximal rank and typical rank have attracted many researchers' interest in this field, however, are still not solved completely. Typical rank is important since tensors can not have ranks other than typical ranks with probability 1. Recently authors defined a class of absolutely nonsingular 3-tensors and absolutely full column rank 3-tensors and proved that the existence of an absolutely nonsingular tensor or a full column rank tensor is essential for determining a typical rank of a special size of 3-tensors. Absolute nonsingularity and absolutely full column rank of a 3-tensor are verified by the positivity of their determinant polynomial. There are several algebraic methods to verify the positivity of a multivariate polynomial, for example, Polya's method. In this talk, we discuss about reduction of verification of positivity of a multivariate determinant polynomial to one of positivity of a single variate function, which makes the problem more tractable. We claim the equivalence of Cukierman's characteristic function and Canny's one and hence reduces the calculation of Cukierma's characteristic function to the calculation of a multivariate resultant. Furthermore, we show examples of verification of absolutely nonsingular tensors through Cukierman's theorem. A direction of further researches is also commented.

Keywords: Typical rank; absolutely nonsingular 3-tensors; absolutely full column rank 3-tensors; positivity of determinant polynomials; positivity of characteristic functions.

1. Introduction

In tensor data analysis, maximal rank and typical rank have attracted many researchers' interest (Atkinson, M.D.& Stephens, M.(1979), Comon P., et al. (2009), and Sumi et al. (2010)). However, determination of them are still not solved completely. An $n \times n \times p$ 3-tensor can be written as a slice of $n \times n$ matrices, $T = (A_1; A_2; \ldots; A_p)$. T is said to be absolutely nonsingular if $\sum_{i=1}^{n} x_i A_i$ is nonsingular for all $\boldsymbol{x} = (x_1, \ldots, x_p) \neq \mathbf{0}$. A homogeneous polynomial of degree n in $\boldsymbol{x}$

$$f_T(\boldsymbol{x}) = \det\left(\sum_{i=1}^{p} x_i A_i\right)$$

is called a determinant polynomial of the tensor T (see Sakata et al. (2009)). Now the following is obvious.

Proposition 1 *The absolute nonsingularity of $T = (A_1; A_2; \cdots; A_p)$ is equivalent to the nonzero property of the determinant polynomial $f_T(\boldsymbol{x})$, where the nonzero property means that $f_T(\boldsymbol{x})$ does not vanish for all $\boldsymbol{x} \neq \mathbf{0}$.*

This was extended to that an $n \times m \times p$ tensor is said to be absolutely full column rank if $\sum_{i=1}^{n} x_i A_i$ is of full column rank for all $\boldsymbol{x} = (x_1 \ldots, x_p) \neq \mathbf{0}$. Then it holds

Proposition 2 *A homogeneous polynomial in $\boldsymbol{x}$*

$$f_T(\boldsymbol{x}) = \det\left(\left(\sum_{i=1}^{p} x_i A_i\right)\left(\sum_{i=1}^{p} x_i A_i\right)^T\right),$$

is nonzero property if and only if T is absolutely full column rank.

Authors (Sumi et al., 2013 & 2014) found that the existence of absolute nonsingularity of full column rank tensors and absolutely full column rank tensors are closely related to determination of typical ranks of a certain size of tensors. In this paper we discuss a computational problem around this topic.

2. Reduction to a single variate polynomial from a multivariate polynomial

It is easy to see that non-zero property is replaced by positivity. Then, various methods, for example, Polya's criterion, exist for proving positivity of multivariate polynomials (see Lasserre, J.B. (2010), Loera, J.A., & Santos, F. (2001), Nie, et al. (2005), Putinar, M. (1993), and Schmudgen, K. (1991)). However, Cukierman, F. (2007) defined a characteristic function and proved the main theorem of Cukierman, which is directly useful to verification of absolutely nonsingularity of a tensor. We state his theorem after several definitions. Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$.

Definition 3 $\mathbb{K}(n,d)$ *denotes the set of homogeneous polynomials in n variables $x_1,\dots,x_n$ with coefficients in $\mathbb{K}$.*

Definition 4 *For $\mathbb{K}=\mathbb{R}$, $F(\boldsymbol{x})\in\mathbb{R}(n,d)$ is said to be positive if $F(\boldsymbol{x})>0$ for all $\boldsymbol{x}\neq\mathbf{0}$.*

Definition 5 $\nabla(n,d,\mathbb{C})\subset\mathbb{C}(n,d)$ *denotes the set of singular homogeneous polynomials of degree d in n variables. That is, $\nabla(n,d,\mathbb{C})=\{F\in\mathbb{C}(n,d)|\frac{\partial F}{\partial x_i}(\boldsymbol{x})=0,\forall i,\exists\boldsymbol{x}\in\mathbb{C}^n-\mathbf{0}\}$.*

Proposition 6 *There is a polynomial Δ with rational coefficients called a discriminant such that*

$$\nabla(n,d,\mathbb{C})=\{F\in\mathbb{C}(n,d)|\Delta(F)=0\},$$

and it is uniquely determined up to a multiplicative constant.

Note that any homogeneous polynomial $F\in\mathbb{K}(n,d)$ is expressed as

$$F=\sum_{|\lambda|=d}F_\lambda\boldsymbol{x}^\lambda \textit{ with } F_\lambda\in\mathbb{K},$$

and that $\Delta(F)=0$ is used in the sense of $\Delta(\{F_\lambda\})=0$. Cukierman normalize Δ such that $\Delta(J)=1$ where

$$J(\boldsymbol{x})=\sum_{1\le j\le n}x_j^d.$$

Then he define

Definition 7

$$\begin{aligned}\nabla &= \nabla(n,d,\mathbb{R})=\nabla(n,d,\mathbb{C})\cap\mathbb{R}(n,d)\\ &= \{F\in\mathbb{R}(n,d)|\frac{\partial F}{\partial x_i}(\boldsymbol{x})=0,\forall i,\exists\boldsymbol{x}\in\mathbb{C}^n-\mathbf{0}\}\\ &= \{F\in\mathbb{R}(n,d)|\Delta(F)=0\},\end{aligned}$$

and

Definition 8 *For $F\in\mathbb{K}(n,d)$, the characteristic polynomial of F with respect to J is defined by*

$$\chi(F;J)(t)=\Delta(F+tJ)\in\mathbb{K}[t],$$

where

$$J(\boldsymbol{x})=\sum_{1\le j\le n}x_j^d.$$

The following is the main theorem of Cukierman, F. (2007) and directly useful to verification of absolutely nonsingularity of tensors.

Theorem 9 *(Cukierman) Let $F \in \mathbb{R}(n,d)$. Then $\chi(F;J)(t) > 0 \ \ \forall t \in \mathbb{R}_{\geq 0}$ if and only if F is positive.*

From above Theorem 9, what we need is to know how to calculate the characteristic function.
Note that in Cukierman, F. (2007) did not discussed the calculation of his characteristic function.

3. A brief review of resultants
Let n homogeneous polynomial $f_0, \ldots, f_1$ with degrees $d_0, \ldots, d_n$ be defined over an algebraically closed field k. The resultant $Res_{d_0,\ldots,d_n}(f_0, \ldots, f_n)$ is an irreducible polynomial in the coefficient of $f_0, \ldots, f_n$. It vanishes whenever $f_0, \ldots, f_n$ have a common root in the projective space. The study of resultants goes back to Sylvester, Bezou, Cayley, Macaulay and Dixon.
Macaulay Formula
For multivariate resultants we refer to the book by Cox et al. (1998). We consider a system of $n+1$ homogeneous polynomial equations of $n+1$ variables, $x_0, x_1, \ldots, x_n$,

$$f_0(x_0, \ldots, x_n) = 0, f_1(x_0, \ldots, x_n) = 0, \ldots, f_n(x_0, \ldots, x_n) = 0.$$

Multivariate resultants tell us when the system of equations has the non-trivial solutions. We have

Theorem 10 *For universal polynomials $f_0, \ldots, f_n$*

$$Res = \det(M_n)/\det(M'_n) \ if \ \det(M'_n) \neq 0.$$

Note that M_n and M'_n are matrices defined in the page 104 of Cox et al. (1998). We don't cite them explicitly since they are quite complicate. This theorem is due to Macaulay, F. (1902). The method to calculate matrices M_n and M'_n is also explained in details in Cox et al. (1998). For the case of $\det(M'_n)$, Canny, J. (1990) introduced

$$Res(f_0 - tx_0^{d_0}, f_1 - tx_1^{d_1}, \ldots, f_n - tx_n^{d_n}),$$

where $d_0, d_1, \ldots, d_n$ are the total degree of f_i. Then

$$cf(t) = Res(f_0 - tx_0^{d_0}, f_1 - tx_1^{d_1}, \ldots, f_n - tx_n^{d_n}) = \frac{det(M_n - tI)}{det(M'_n - tI)}.$$

From the construction we claim

Proposition 11 *If $f_0 = f_1 = \cdots = f_n$ have the same total degree d, the generalized characteristic function $\chi(t)$ of Cukierman is the same as the generalized characteristic function $cf(t)$ of Canny with a minor change $t \to -dt$.*

4. Examples
From Proposition 11 we can calculate the characteristic function $\chi(t)$, by Macaulay and Canny method, whose positivity verify absolutely nonsingularity of tensor. We implemented the program of calculating multivariate resultant by following the chapter 3 of Cox et al (1998). Here we show the calculation of characteristic functions below.

```
Example 1:T1
A;
[ 1 0 0 0 ]
[ 0 1 0 0 ]
[ 0 0 1 0 ]
[ 0 0 0 1 ]
[112] B;
[ -1 0 1 1 ]
```

```
[ 1 -1 1 -1 ]
[ -1 -1 -1 -1 ]
[ 0 1 1 0 ]
[113] C;
[ 0 1 -1 1 ]
[ 0 -1 1 1 ]
[ 0 -1 0 1 ]
[ -1 -1 1 1 ]
[114] F0(determinant polynomial);
x^4-3*y*x^3+(7*y^2-2*z*y+z^2)*x^2+(-8*y^3+4*z*y^2-4*z^2*y-z^3)*x+6*y^4-6*z*y^3
+12*z^2*y^2+7*z^3*y+2*z^4
[115] Cukierman's Characteristic function;
t^27+324*t^26+49536*t^25+4755456*t^24+321818112*t^23+
16340908032*t^22+647081410560*t^21+20506794983424*t^20
+529672229683200*t^19+11298694177751040*t^18+200993550593163264*t^17
+3003040780555124736*t^16+37876475135979945984*t^15+404640294638614216704*t^14+
3668285353184458702848*t^13+28231340094337943863296*t^12
+184258014978343207698432*t^11+1017248687189151379881984*t^10
+4729863221393474786951168*t^9+18402926668741369959284736*t^8
+59370815843289470607556608*t^7+156811189444372808563949568*t^6+333082976048
757388244680704*t^5+554605709395993046361833472*t^4+696603086736664564296843264*t^3
+620153046233625278413799424*t^2+348563773058712100279418880*t+
92950339482323226741178368
```

Clearly T_1 has the positive characteristic function on $t \geq 0$ and it is absolutely nonsingular.

```
Example 2:T2
A;
[ 1 0 0 0 ]
[ 0 1 0 0 ]
[ 0 0 1 0 ]
[ 0 0 0 1 ]
[112] B;
[ 0 1 -1 0 ]
[ 0 1 1 -1 ]
[ 1 -1 -1 0 ]
[ 1 0 1 0 ]
[113] C;
[ -1 1 -1 1 ]
[ 1 -1 -1 -1 ]
[ 0 1 0 -1 ]
[ -1 0 1 -1 ]
[114] F0(determinant polynomial);
x^4-3*z*x^3+(y^2-z*y+5*z^2)*x^2+(2*y^3+4*z*y^2+2*z^2*y-7*z^3)*x+3*y^4+
4*z*y^3-5*z^2*y^2-6*z^3*y+6*z^4
[115] Cukierman's Characteristic function;
t^27+360*t^26+61488*t^25+6630528*t^24+506778624*t^23+29218480128*t^22+
1320545796096*t^21+47997726621696*t^20+1428255087132672*t
^19+35240227143417856*t^18+727592959777701888*t^17+12651993826332770304*t^16
+186095259410919063552*t^15+2321336825062040272896*
t^14+24581897899295813664768*t^13+220895602993035044978688*t^12+
1681318447890958796193792*t^11+10802943471138768282451968*t^10+
58298780221616611739566080*t^9+262367266917063028270694400*t^8+
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975150101034669965278445568*t^7+2953954741633096075255480320*t^6
+7161307664991450109977821184*t^5+13539351469197044751557197824*t^4
+19206612763016217552173924352*t^3+192066127630162175521739
24352*t^2+12059966153521811021132464128*t+3573323304747203265520730112
```

It is clear that Cukierman's characteristic function is positive for $t \geq 0$. And so it is absolutely nonsingular.

5. Further works

The following is within the scope of this topics.
(1) Estimate the probability of $n \times n \times 3$ tensors being absolutely nonsingular.
(2) Show non-existence of any $6 \times 6 \times 3$ absolutely nonsingular tensors by this method.

Remark 12 *A single variate polynomial $f(t)$ is positive for $t > 0$ if and only if $f(0) > 0$ and $f(t) > 0$ for all $t > 0$ such that $f'(t) = 0$. This makes possible an automatic search of absolutely nonsingular tensors and leads to an estimate of the probability of absolutely nonsingular tensors.*

Remark 13 *It is known that there isn't absolutely nonsingular tensors of size $6 \times 6 \times 3$ by a theory of bilinear forms. If this fact is proved by the proposed method in this paper, it is quite interesting.*

6. Conclusions

We discussed absolutely nonsingular and absolutely full column rank of 3 tensors.This is verified by the positivity of their determinant polynomial which is a multivariate polynomial. We show the verification of positivity of determinant multivariate polynomials can be reduced to the verification of positivity of a single variate polynomial through Cukierman's theorem. We gave several examples which is verified to be absolutely nonsingular tensor through this reduction, which appeared in Sakata et al. (2011). Further directions of this research were also commented.

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