Layered 2D ray-averaging approximation

Wilfrid S. Kendall*
Department of Statistics, University of Warwick, Coventry, UK - w.s.kendall@warwick.ac.uk

Audrey Kueh
Department of Statistics, University of Warwick, Coventry, UK - a.kueh@warwick.ac.uk

Abstract

In industrial contexts, computerised tomography (CT) involves rotation of the X-ray source around a circle surrounding the object. In the 3D case, there are significant mathematical issues obstructing exact reconstruction. These are partly resolved in practice by the FDK approximation (Feldkamp, Davis, & Kress, 1984). As part of an investigation into the use of CT in statistical quality control, we discuss an alternative and more intuitive approximation.

Keywords: *CT scan; FDK approximation; layered 2D ray-averaging approximation; noise model.*

1. Introduction

This is a progress report on some of the work being carried out by an interdisciplinary team of statisticians and engineers working at the University of Warwick Department of Statistics and Warwick Manufacturing Group. The team is funded by UK EPSRC research grant EP/K031066, and is exploring the use of computerized tomography (CT) for the purposes of quality control in Additive Layer Manufacturing (ALM: commonly known as "3D printing").

CT involves an *object*, represented by a density f which is of compact support and surrounded by a *source curve* along which the X-ray source moves; data are collected by detector screens on the other side of the object. In reality, in industrial contexts, the source remains fixed while the object moves along a curve, but it is convenient to use a rotating frame of reference and to treat the object position as fixed.

CT in an industrial context differs from CT in medicine in a number of significant ways:

1. Whether the inanimate object receives a high radiation dosage from X-rays is not a primary concern;
2. However objects designed for industrial purposes are typically made up of material involving high atomic numbers (steel, titanium, ...), consequently CT scans have to use high energy X-rays and can be relatively slow.
3. In contrast to the helical geometry of source scans often found in medicine, industrial CT typically involves circular source scans.

Point 2 is an issue for ALM quality applications, where the emphasis is often on rapid manufacture. Our team is thus interested in the trade-off between shortness of elapsed time and quantity of statistical information. Point 3 raises significant mathematical issues, as it is known that reconstructions based on a source circle must lead to biases or *artefacts* in any given reconstruction. In practice, engineers use a combination of an ingenious approximation (the FDK approximation mentioned below), and subject-specific knowledge.

So far the team has been investigating the statistical quality of the resulting projections, distortions in the data arising from detailed geometry of the source, and quality issues of the detector screens. In this paper we discuss a question related to point 3, namely an alternative to the usual approximation used in the case of source circles.

2. Layered 2D ray-averaging approximation

We begin by summarizing the standard approach to 2D CT reconstruction. Data are produced by the X-ray transform applied to the target density f, in the form of an ensemble of line integrals

$$(Df)(\ell) \quad = \quad (Df)(\underline{x}, \underline{\theta}) \quad = \quad \int_{-\infty}^{\infty} f \, \mathrm{d}\ell \quad = \quad \int_{-\infty}^{\infty} f(\underline{x} + t\underline{\theta}) \, \mathrm{d}t \tag{1}$$

taken over lines $\ell = \{\underline{x} + t\underline{\theta} : -\infty < t < \infty\}$ (here $\underline{x}$ denotes a location, $\underline{\theta}$ a unit vector defining a direction) which intersect the object. (In practice the integrand in (1) is positive only on at most a half-infinite interval of the line ℓ, so in fact $(DF)(\ell)$ is an integral along a ray.) The projection-slice theorem relates these line integrals to the Fourier transform of f by means of a simple change of variables (Natterer & Wübbeling, 2001, Thm. 2.1): if the unit vector $\underline{\theta}$ is chosen to be perpendicular to $\underline{\xi} \neq 0$ then

$$\hat{f}(\underline{\xi}) \quad = \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\underline{x}) e^{-i\underline{x}\cdot\underline{\xi}} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \quad = \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} (Df)(t\underline{\xi}/|\underline{\xi}|, \underline{\theta}) e^{-it|\underline{\xi}|} \, \mathrm{d}t \,. \tag{2}$$

Recovery of f then follows from inversion of the Fourier transform $\hat{f}$.

In practice reconstruction is often formulated using a filter (to address issues arising from discretization error) and back-projection (adjoint of the operation of projection onto a given line); however here we simply discuss reconstructions in terms of appropriate truncations of the Fourier formulation.

The complication for 3D CT reconstruction is the restriction that typically not all line integrals are available, only those arising from lines intersecting the source curve $\mathcal{C}$, the trajectory along which the X-ray source travels. The Tuy-Kirillov condition (Natterer & Wübbeling, 2001, p. 25; also Natterer, 1986, Section VI.5) asserts that exact reconstruction is possible (at least in principle) when $\mathcal{C}$ transversally intersects every plane which itself intersects the object under study. This holds when $\mathcal{C}$ is a suitably positioned helix, which can be the case for medical applications. However industrial CT scanners are typically based on a source curve which is a circle, and the Tuy-Kirillov condition fails in such 3D cases. Nevertheless Feldkamp, Davis, & Kress (1984) proposed the ingenious *FDK approximation*, commonly used in engineering contexts (using domain knowledge to remedy the inevitable defects of the approximation). Natterer & Wübbeling (2001, Section 5.5.1) summarize the FDK approximation: in essence the idea is to approximate 2D recovery for each horizontal plane of height h by using collections of pencils of lines: such a pencil is made up of those lines passing through a point $\underline{x}$ on the source circle $\mathcal{C}$ and intersecting the horizontal line at height h in the vertical plane through the origin orthogonal to the vector $\underline{x}$.

The FDK approximation has the desirable property of preserving the vertical line integrals of f (Feldkamp, Davis, & Kress, 1984, Appendix A), but reconstructions are subject to blurring and artefacts off the horizontal plane containing the source circle $\mathcal{C}$. In practice these deficiencies are remedied empirically by expert engineering subject knowledge.

In this paper we propose a simpler alternative to the FDK approximation, namely the *layered 2D ray-averaging approximation*. The idea is as follows: for reconstruction of f on the horizontal plane of height h, we would like to use the standard 2D reconstruction procedure in that plane. However the relevant X-ray transforms are not available; when $h \neq 0$ then the horizontal plane (and thus the corresponding lines) does not intersect the source circle. Instead, approximate each X-ray transform by a suitable linear combination of two line integrals involving a pair of *complementary rays*.

The complementary rays are defined as follows: if ℓ is the target line in the horizontal plane of height h, then let $\ell^{\downarrow}$ be the projection of ℓ onto the plane containing $\mathcal{C}$. Let A_1 and A_2 be the two points of intersection of $\ell^{\downarrow}$ with $\mathcal{C}$, and let B be the midpoint of the segment A_1A_2. The two complementary rays are defined by ℓ_1 and ℓ_2 and start at A_1 and A_2 respectively, and both pass through the point $B^{\uparrow}$ which lies in the plane at height h and directly above B. Figure 1 illustrates the construction.

The relevant linear combination is given by the following expression for the layered 2D ray-averaging approximation:

$$(Df)(\ell) \quad \approx \quad \frac{|A_1A_2|}{\sqrt{4h^2 + |A_1A_2|^2}} \frac{(DF)(\ell_1) + (DF)(\ell_2)}{2} \,. \tag{3}$$

The multiplier $|A_1A_2|/\sqrt{4h^2 + |A_1A_2|^2}$ compensates for the slopes of ℓ_1 and ℓ_2 relative to the horizontal plane.

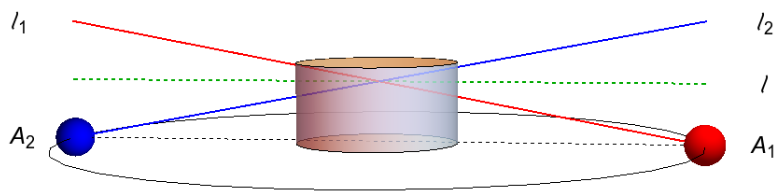

Figure 1: Illustration of the complementary rays construction.

Evidently this approximation leads to distortion of the reconstruction, just as in the case of the FDK approximation. However its simplicity allows us to quantify the distortions usefully. We do not propose this as an alternative to the FDK approximation, but rather as a useful proxy to gain insight into the biases arising from use of FDK.

3. Error Analysis

The principal quantification of bias arises from a simple second-order Taylor series approximation in the vertical coordinate. Suppose that the density f is twice continuously differentiable in the vertical direction. Changing variables of integration appropriately, and writing $S = |A_1 A_2|$, $\underline{y} = (A_2 - A_1)/S$, and $\underline{z}$ for a unit vertical vector,

$$\begin{aligned}\frac{S}{\sqrt{4h^2+S^2}}(DF)(\ell_1) &= \int_0^S f(A_1 + t\underline{y} + 2(t/S)h\underline{z})\,\mathrm{d}t \\ &= \int_0^S f(A_1+t\underline{y}+h\underline{z})\,\mathrm{d}t + h\int_0^S \frac{2t-S}{S}\, f_3(A_1+t\underline{y}+h\underline{z})\,\mathrm{d}t + \frac{h^2}{2}\int_0^S \left(\frac{2t-S}{S}\right)^2 f_{33}(A_1+t\underline{y}+\zeta_1(t)h\underline{z})\,\mathrm{d}t\,.\end{aligned}$$

Here f_3 and f_{33} are the first and second vertical derivatives of f, while $\zeta_1(t) \in (-t, t)$ is chosen to ensure the equality holds for this second-order Taylor expansion about h. The first integral is exactly $(Df)(\ell)$. Averaging with a corresponding expression based on $(DF)(\ell_2)$, the middle integrals cancel out, hence leading to the explicit error bound

$$\left|(Df)(\ell) - \frac{S}{2\sqrt{4h^2+S^2}}\left((DF)(\ell_1) + (DF)(\ell_2)\right)\right| \quad\leq\quad \frac{h^2 S}{3}\sup\{|f_{33}|\}\,. \tag{4}$$

(Actually a tighter bound is available when the support of f is contained in a vertical centred cylinder of radius $r < R$.) In particular the layered 2D ray-averaging approximation is exact when the vertical variation of the density f is linear, and is otherwise controlled by $\sup\{|f_{33}|\}$, a measure of the vertical smoothness of the density.

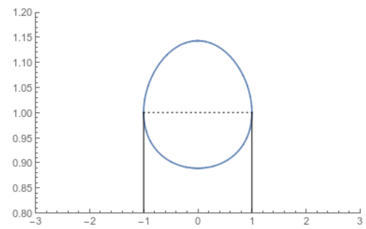

Figure 2:
Section of the ovoid region within which the artefact occurs for a horizontally capped vertical cylinder, with unit radius base centred at the centre of the source circle $\mathcal{C}$. The dotted line marks the upper interface of the capped cylinder. Here the source circle radius is given by $R = 8$.

On the other hand, and in contrast to the FDK approximation, a computation shows this approximation does not preserve the exact value of the vertical integrals.

4. Interfaces: a case study

The error analysis given in the previous section controls the bias of the layered 2D ray-averaging approximation when the object has vertically smooth mass density, but does not apply if the object has any interface discontinuities which do not include a strictly vertical orientation. It is useful to investigate these biases (also called artefacts) for a simple test object. Accordingly we have carried out a case study for the test object which is a vertical cylinder capped horizontally at level 0 and level 1, with base a circle of radius $r = 1$, and centred at the origin (which is the centre of the source circle $\mathcal{C}$, itself of radius $R = 8 > r$). The cylinder is taken to be of unit intensity, while the exterior is of zero density. The interface at level 0 causes no difficulty; however the interface at level h gives rise to an artefact.

The configuration has been chosen to be rotationally symmetric around the vertical axis; it is therefore possible to compute the resulting artefact exactly. In particular, simple geometric arguments allow exact computation of the X-ray transform. Considering all lines at height h, it can be seen that neither of the approximating rays ℓ_1 or ℓ_2 from (3) will intersect the cylinder if $h < 0$ or $h > R/(R-r)$, while the intersections of both will coincide with their intersections with the non-capped cylinder if $0 \leq h \leq R/(R+r)$. Indeed a geometric argument shows that the intersections of ℓ_1 and ℓ_2 will fail to coincide with their intersections with the non-capped cylinder when the absolute distance y from the origin of the horizontal projection $\ell^{\downarrow}$ exceeds a particular function of h.

In fact the "hole" theorem (Natterer, 1986, Section II.3) can be used to show that the artefact must be contained within the ovoid region depicted in section in Figure 2. In principle the actual density function of the artefact region could be computed exactly, exploiting the radial symmetry of the case study. However it is easiest to determine the density function using numerical techniques for 2D reconstruction: Figure 3 depicts an illustrative example; a density plot of a section of the artefact through the vertical axis of symmetry. It can be seen that the artefact region does indeed form an ovoid region, with density intermediate between that of the cylinder and the zero density of the exterior.

Similar calculations can be carried out for a region which is a ball centred on the axis of radial symmetry. The resulting artefact corresponds again to a vertical distortion; Figure 4 provides an illustrative example.

5. Conclusions

We have discussed an alternative to the FKD approximation, giving a rather simpler approximation for 3D reconstruction based on layered 2D ray-averaging approximation. Explicit bounds can be obtained for the approximation error in the case of vertically smooth density. A case study for the effect of interface discontinuity is described, and artefacts are explicitly computed. The purpose of this alternative is not to compete with the FKD approximation, but to provide insight into the nature of the bias that arises from such approximations.

Future work will include: variation of the approximation by modification of the definition of complementary rays; also examination of non-radially symmetric cases (in particular, cylinders and balls not centred on the

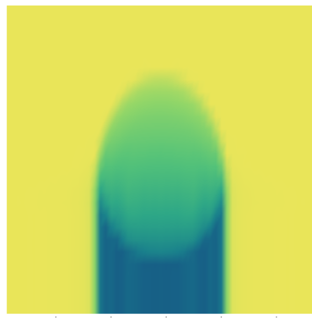

Figure 3:
Density plot of a section of the artefact for the horizontally capped vertical cylinder discussed in Figure 2.

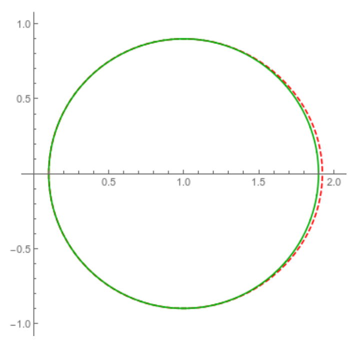

Figure 4:
Graph of the region within which the artefact occurs for a ball centred on the axis of radial symmetry of the source circle $\mathcal{C}$. In this case the axis of radial symmetry runs across the page. Here sphere radius is 0.9 while source circle radius is 8.

axis of radial symmetry). It would be of considerable interest to derive perturbation calculations to describe the behaviour of artefacts for objects which are close to this axis. More speculatively, information concerning artefacts arising from non-horizontally capped cylinders would be of great value: our collaborators inform us that users of FKD deliberately tilt their structures so as not to exhibit any horizontal interfaces: it would be illuminating to obtain criteria for the extent to which this practice would reduce artefacts in the case of layered 2D ray-averaging approximation.

Statistical investigation is of course not solely about bias, and further work is required on suitable noise models. Other empirical studies carried out by our team have established mean-variance models for the signal arising from CT projections in the particular regime that is suited to ALM using material with high X-ray opacity. There is an interesting question about what sort of correlation is to be expected. It seems reasonable to expect a white-noise model to be appropriate for modelling detector noise (which is relevant when considering individual projections). It is possible that relevant correlations might appear between different projections, arising from small inhomogeneities in the structure under investigation. A possible noise model is suggested by the integral-geometric context of CT. Suppose such inhomogeneities arise as superpositions of independent perturbations, each perturbation being given by adding a small increment to material density on one side of a random plane, and subtracting it on the other. There is a natural geometric link to the mechanism of CT, arising *via* line/plane duality. The resulting noise has a Gaussian limit, which is a variant on the classic multidimensional random field known as Lévy's Brownian process (see for example Chentsov, 1957). It seems likely that this would produce correlations which are too long-range for the present application, but some modification of this noise model might prove useful.

References

Chentsov, N. (1957). Lévy Brownian Motion for Several Parameters and Generalized White Noise. *Theory of Probability & Its Applications*, 2(2), 265-266.

Feldkamp, L., Davis, L., & Kress, J. (1984). Practical cone-beam algorithm. *Journal of the Optical Society of America*, 1(6), 612–619.

Natterer, F. (1986). *The mathematics of computerized tomography* (222 pp). Stuttgart/Chichester, Teubner/Wiley.

Natterer, F., & Wübbeling, F. (2001). *Mathematical Methods in Image Reconstruction* (226 pp). Philadelphia, PA, Society for Industrial and Applied Mathematics.