Empirical likelihood confidence intervals in the presence of unit non-response

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Abstract

Suppose that the parameter of interest is a complex parameter defined as a solution of an estimating equation (Godambe, 1960). For example, this parameter could be a regression parameter, a quantile, a ratio or a mean. The aim is to estimate, test and construct a confidence interval for this parameter. We assume that the data are based on a stratified sample of units selected with unequal probabilities. Suppose that we have unit non-response according to a uniform response mechanism within cells; that is, we assume that any units can be missing with the same response probability within cells. We consider a reverse approach (Fay, 1991); that is, the response mechanism is the first phase and the second phase is a stratified unequal probability sampling design with negligible sampling fraction. We show how the empirical likelihood approach proposed by Berger and De La Riva Torres (2015) can be used to estimate complex parameters and to construct confidence intervals in the presence of unit non-response. The proposed estimator is based on response rates estimated within cells. The proposed empirical likelihood confidence interval takes into account of (i) the design, (ii) the response mechanism, (iii) the randomness of the estimator of the response rates and (iv) the population level information. This confidence interval does not rely directly on the normality of the point estimator and does not require variance estimation, linearisation, re-sampling and estimation of a design effect.

Keywords: design-based inference, estimating equations, empirical likelihood, stratification, unequal inclusion probabilities.

1. Introduction

Let $U$ be a finite population of $N$ units; where $N$ is a fixed quantity which is not necessarily known. Suppose that the population parameter of interest $\theta_N$ is the unique solution of the following estimating equation (Godambe, 1960).

$$G(\theta) = 0, \quad \text{with} \quad G(\theta) = \sum_{i \in U} g_i(\theta);$$

where $g_i(\theta)$ is a function of $\theta$ and of the values of variables for unit $i$.

Let $s \subset U$ be a sample of sample of size $n$. Consider the following empirical log-likelihood function

$$\ell(m) = \log \left( \prod_{i \in s} m_i \right) = \sum_{i \in s} \log(m_i), \quad (1)$$

where $\prod_{i \in s}$ and $\sum_{i \in s}$ denote the product and the sum over the sampled units and the $m_i$ are scale loads (Hartley and Rao, 1968). The $m_i$ can be estimated by the values $\hat{m}_i$ which maximise $\ell(m)$ subject to the constraints $m_i \geq 0$ and

$$\sum_{i \in s} m_i c_i = C; \quad (2)$$

where $c_i$ is a $Q \times 1$ vector associated with the $i$-th sampled unit and $C$ is a $Q \times 1$ vector. Berger and De La Riva Torres (2015) showed that the $\hat{m}_i$ are given by

$$\hat{m}_i = \left( \pi_i + \eta^\top c_i \right)^{-1}, \quad (3)$$
The quantity $\eta$ is such that the constraint (2) holds.

The vectors $c_i$ and $C$ satisfied the regularity conditions given by Berger and De La Riva Torres (2015). These vectors contains the information about the design (inclusion probabilities and stratification) and the population level information (for more details see Berger and De La Riva Torres, 2015). For example, when $c_i = N\pi_i^{-1}$ and $C = N$, the $\hat{m}_i$ are Horvitz and Thompson (1952) weights: $\hat{m}_i = \pi_i^{-1}$.

Let the $\hat{m}_i^*(\theta)$ be the values which maximise (1) subject to the constraints $m_i \geq 0$ and

$$\sum_{i \in s} m_i c_i^* = C^*$$

with $c_i^* = (c_i^T, g_i(\theta))^T$ and $C^* = (C^T, 0)^T$, for a given $\theta$. Let $\ell(\hat{m}^*, \theta) = \sum_{i \in s} \log(\hat{m}_i^*(\theta))$ be the maximum value of (1) under the constraint (4). The empirical log-likelihood ratio function is defined by the following function of $\theta$.

$$\hat{r}(\theta) = 2 \{ \ell(\hat{m}) - \ell(\hat{m}^*, \theta) \}.$$ (5)

The maximum empirical likelihood estimate $\hat{\theta}$ of $\theta_N$ is defined by the unique value of $\theta$ which minimises the function (5) (or maximise the empirical likelihood function $\ell(\hat{m}^*, \theta)$). Note that when $c_i = N\pi_i^{-1}$, $C = N$ and $g_i(\theta) = y_i - n^{-1}\theta\pi_i$, we have $\hat{m}_i = \pi_i^{-1}$ and $\hat{\theta}$ is the Horvitz and Thompson (1952) estimator $\hat{\theta} = \sum_{i \in s} y_i\pi_i^{-1}$. When $g_i(\theta) = y_i - \theta N^{-1}$, $\hat{\theta}$ is the Hájek (1971) ratio estimator $\hat{\theta} = N\sum_{j \in s} \pi_j^{-1})^{-1} \sum_{i \in s} y_i\pi_i^{-1}$.

Berger and De La Riva Torres (2015) showed that $\hat{r}(\theta_N)$ follows asymptotically a chi-squared distribution with one degree of freedom. Hence a confidence interval for $\theta_N$ is given by

$$\{ \theta : \hat{r}(\theta) \leq \chi^2_1(\alpha) \};$$ (6)

where $\chi^2_1(\alpha)$ is the upper $\alpha$-quantile of the chi-squared distribution with one degree of freedom.

2. Unit Non-response

Consider a uniform response mechanism where all the units respond independently with the same response probability $p_r$. Let $r_i$ be the response indicator: $r_i = 1$ if the unit $i$ is a respondent and $r_i = 0$ otherwise. We propose to substitute $g_i(\theta)$ by $r_i g_i(\theta)$. The maximum empirical likelihood estimate $\hat{\theta}$ of $\theta_N$ is still defined by the value of $\theta$ which minimises the function (5).

When the sampling fraction is negligible, it can be shown that the empirical log-likelihood ratio function $\hat{r}(\theta_N)$ follows asymptotically a chi-squared distribution with one degree of freedom, under a two-phase reverse approach (Fay, 1991). The first phase is the response mechanism and the second phase is the sampling design. Hence, a confidence interval for $\theta_N$ is given by (6).

The uniform response assumption is often unrealistic in practice. It is common practice to form a finite number of re-weighting cells and to assume uniform response within cells. We will show how the proposed approach can be extended in this case.

3. Simulation

We give here the results of simulation study which could be found in Berger and De La Riva Torres (2015) for a single re-weighting cell and under a uniform response mechanism. The 1998-1999 British Family Expenditure Survey data were duplicated three times to create an artificial population of $N = 19,890$ households. Samples of size $n = 500$ are selected with unequal probabilities proportional to the first-order inclusion probabilities given in the dataset. The parameters of interest are $q$-quantiles of the equivalent total weekly household expenditure (Department of Social Security, 2001). The population level information is the numbers of individuals within age-sex groups (0-19, 20-39, 40-59, 60+). Non-responding households are generated.
according to a uniform response mechanism with the average response rates 60%, 70% and 80%. We consider confidence intervals with the nominal level of 95%. The observed coverages and tail error rates are given in Table 1. Most of the coverages and tail error rates are not significantly different from 95% and 2.5%. With auxiliary variables, the upper tail error rates can be slightly larger than 2.5%.

5. Conclusions

Standard confidence intervals based on the central limit theorem require the normality of the point estimator and variance estimates which often involve linearisation or re-sampling. The proposed method does not rely on variance estimates, linearisation or re-sampling, even if the parameter of interest is not linear. The proposed method does not rely directly on the normality of the point estimator. Empirical likelihood confidence intervals can be easier to compute than standard confidence regions based on variance estimates. It provides a less computationally intensive alternative to bootstrap. The proposed approach naturally includes population level information and the information about the sampling design. Note that the proposed confidence interval takes into account of the effects of the nonresponse, the population level information and the sampling design.

Table 1: Empirical likelihood coverages (%) for quantiles $Y_q$. Unit nonresponse. FES data. Source: Berger and De La Riva Torres (2015).

<table>
<thead>
<tr>
<th></th>
<th>Av. resp. rate = 60%</th>
<th>Av. resp. rate = 70%</th>
<th>Av. resp. rate = 80%</th>
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<tr>
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<td>$q$ Overall Lower Upper</td>
<td>$q$ Overall Lower Upper</td>
<td>$q$ Overall Lower Upper</td>
</tr>
<tr>
<td>Without</td>
<td>10% 95.2 2.1 2.7</td>
<td>95.0 2.6 2.4</td>
<td>94.8 2.4 2.8*</td>
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<tr>
<td>population</td>
<td>25% 94.9 2.2 2.9*</td>
<td>94.8 2.4 2.8</td>
<td>94.8 2.5 2.6</td>
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<tr>
<td>level</td>
<td>50% 95.1 2.3 2.7</td>
<td>95.0 2.3 2.7</td>
<td>95.3 2.1* 2.5*</td>
</tr>
<tr>
<td>information</td>
<td>75% 94.5* 2.4 3.1*</td>
<td>94.8 2.2 2.9*</td>
<td>95.4 1.9* 2.7*</td>
</tr>
<tr>
<td>With</td>
<td>10% 94.4* 2.8 2.8*</td>
<td>94.6* 2.6 2.7*</td>
<td>94.8* 2.4 2.8*</td>
</tr>
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</tbody>
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* Coverages (or tail error rates) significantly different from 95% (or 2.5%). p-value ≤ 0.05.

References


