Some remarks on tests of multivariate normality based on measures of shape

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Abstract

There are many tests of multivariate normality and rules for the construction of their statistics. Usually, five basic classes are mentioned. This division results from the way the test statistics are constructed. There are tests based on: randomization rule, measures of shape, union rule and Roy's intersection, multivariate geometry notions, variable transformation. The multivariate measures of shape are useful as statistics carrying some characteristic of the multivariate sample and as a basis for tests of multivariate normality. In the paper an investigation of some tests of multivariate normality based on the measures of skewness and kurtosis is presented.

Keywords: multivariate normality, statistical test, measure of shape, skewness, kurtosis.

1. Introduction

Let us denote by $\boldsymbol{x}$ a p-dimensional random vector and by $\Phi_p(\boldsymbol{x}; \mu, \sum)$ the cdf of the p-dimensional normal distribution, where x$\in \Re^p$ is a point in $\Re^p$, and μ and $\sum$ denote, respectively: p-dimensional expectation vector and pxp covariance matrix. The notation $\boldsymbol{x} \sim \Phi_p(\mu, \sum)$ means that the random vector x follows the distribution given by the cdf (cumulative distribution function) $\Phi_p(\boldsymbol{x}; \mu, \sum)$.

Let HSMN (simple hypothesis multivariate normality) denote a simple hypothesis of the form:

$$HSMN: F_p(\boldsymbol{x}) = \Phi_p(\boldsymbol{x}; \mu_0, \textstyle\sum_0) \tag{1}$$

which means that $F_p(\boldsymbol{x})$ is the cdf of the $N_p(\mu_0, \sum_0)$ distribution where μ_0 and $\sum_0$ are prefixed parameters. In particular, let us assume that $HSMN^*$denotes a hypothesis of the form:

$$HSMN^*: F_p(\boldsymbol{x}) = \Phi_p(\boldsymbol{x}; \mathbf{0}, \mathbf{I}) \tag{2}$$

Which means that $F_p(\boldsymbol{x})$ is the cdf of the $N_p(\mathbf{0}, \mathbf{I})$distribution. Moreover, let us assume that $HCMN$ (composite hypothesis multivariate normality) denotes the hypothesis:

$$HCMN: F_p(\boldsymbol{x}) = \Phi_p(\boldsymbol{x}; \mu, \textstyle\sum) \tag{3}$$

i.e. $F_p(\boldsymbol{x})$ is the cdf of the $N_p(\mu, \sum)$ distribution with unknown parameters μ and $\sum$.

2. Tests based on the measures of skewness and kurtosis

Let us consider two popular measures of multivariate skwness and kurtosis:

- based on Mahalanobis distance – developed by Mardia (1970), (see also Dufor et al (2003)).

The sample statistic for skewness is given by the following formula:

$$b_{1,p} = \frac{1}{n^2} \sum_{i,j=1}^{n} \left\{ (X_i - \bar{X})' S^{-1} (X_j - \bar{X}) \right\}^3 = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij}^3 \tag{4}$$

Analogous sample statistic for kurtosis has the form:

$$b_{2,p} = \frac{1}{n}\sum_{i=1}^{n}\{(X_i - \bar{X})' S^{-1}(X_i - \bar{X})\}^2 = \frac{1}{n}\sum_{i=1}^{n} d_{ii}^2 \qquad (5)$$

where $X_i, \ldots X_n$ constitute n observations, $\bar{X}$ and S are unbiased estimators of, respectively μ and $\sum$, and d_{ij} are the elements of matrix $D = [d_{ij}]$ defined by:

$$D = (x_i - \bar{x})' S^{-1}(x_j - \bar{x}) \qquad (6)$$

The Mardia's measures are invariant. Mardia (1970) proposed tests of normality based on the statistic of the form:

$$M_1 = \frac{N}{6} b_{1,p} \qquad (7)$$

Statistic (7) on assuming hypotheis (3) true, follows the limiting distribution χ^2 with f degrees of freedom where $f = \frac{p(p+1)(p+2)}{6}$. For $p > 7$ one can use the approximation

$$\sqrt{2M_1} - \sqrt{2f - 1} \qquad (8)$$

For large samples the test for the hypothesis $\beta_{1.p} = 0$ requires the determination of the statistic M_1 with rejection of the null hypothesis when $M_1 > \chi^2_{\alpha,f}$ for $p \leq 7$. For $p > 7$ H_{01} is rejected when $\sqrt{2M_1} - \sqrt{2f - 1} > u_\alpha$, where $\Phi(u_\alpha) = 1 - {}^{\alpha}/_{2}$, and $\Phi(u)$ is the standard normal cdf.

Another statistic is given by the formula:

$$M_2 = \frac{N^*\left(b_{2,p} - p(p+2)\right)^2}{8p(p+2)} \qquad (9)$$

Statistic (9), on assuming hypothesis (3) true follows the limiting χ^2 distribution with 1 degree of freedom.

Another statistics have the forms:

$$A2 = \frac{\{b_{2,p} - p(p+2)^*(n-1)\}/(n+1)}{\sqrt{(8p(p+2))/n}} \qquad (10)$$

$$A'2 = \frac{b_{2,p} - E(b_{2,p})}{D(b_{2,p})} \qquad (11)$$

where $D(b_{2,p}) = \sqrt{D^2(b_{2,p})}$, $E(b_{2p}) = \frac{g(n-1)}{n+1}$ and $D^2(b_{2p}) = \frac{8g(n-3)(n-p-1)(\quad)n-p+1}{(n+1)^2(n+3)(n+5)}$

Statistcs (10) and (11) follow, on strength of the central limit theorem, the standard normal $N(0,1)$.

Apart from the directional tests a popular test proposed by Mardia (1970) was the following omnibus test:

$$MSK = M1 + M2 \qquad (12)$$

where statistics $M1$ and $M2$ are determined, respectively, by the statistics (7) and (9). Statistic MSK follows the χ^2distribution with f degrees of freedom, where $f = (n/6)(n+1)(n+2) + 1$.

3. Examination of test power on the basis of theoretical distributions

There are many tests of multivariate normality and the rules of constructing their test statistics. This raises a natural question on which of them are best in the sense of power, which possess the omnibus test property, which are directed tests and which of them are best to be used in practice. The answers to these questions are searched in simulation Monte Carlo experiments.

In the experiment 10000 repetitions of the multivariate normal distribution were considered. This distribution was generated following Wieczorkowski and Zieliński („Komputerowe generatory liczb losowych” p. 105), for n=20,30,40,50,60,70,80,90,100,110, 120; p=2,3,4; α=0,05.

The results received are presented in table 1. The numbers in bold mean that the test based on the given statistic, on assuming true the null hypothesis stating that the multivariate distribution is normal, for a fixed *n, p* and alfa surpasses the assumed significance level. Looking at the numbers in table 1 let us observe that the test sizes deviate from the assumed significance level which may lead to false conclusions. In connection with this, the empirical critical values for the tests investigated were found (see Domański, Wojek (2009)).

Table 1. Tests sizes of multivariate normal distribution for p = 2,3,4 α=0,05;
n=20,30, 40, 50, 60, 70, 80, 90,100, 110, 120.

Test statistics	Sample size (n)										
	20	30	40	50	60	70	80	90	100	110	120
1	2	3	4	5	6	7	8	9	10	11	12
	p=2										
Ml	0.0137	0.0239	0.0306	0.0352	0.0356	0.0392	0.0414	0.0431	0.0443	0.0425	0.0455
M2	0.002	0.0079	0.0141	0.0172	0.0214	0.028	0.0265	0.0279	0.0306	0.0322	0.0347
MSK	0.0089	0.0191	0.0254	0.0293	0.0306	0.0342	0.0362	0.0383	0.0398	0.0385	0.0418
A2	0.0027	0.008	0.0129	0.0161	0.0179	0.0251	0.0234	0.0252	0.0286	0.0277	0.0319
A2'	0.0349	0.0314	0.0335	0.0333	0.0362	0.0402	0.0383	0.0388	0.041	0.0397	0.0441
	p=3										
Ml	0.0076	0.0169	0.0263	0.0329	0.0344	0.03761	0.0405	0.0395	0.0398	0.0431	0.0425
M2	0.0335	0.0362	0.035	0.0397	0.0379	0.0422	0.0402	0.0446	0.0411	0.0447	0.0431
MSK	0.0052	0.0132	0.0217	0.0281	0.0302	0.0341	0.0389	0.0355	0.0373	0.0398	0.0394
A2	0.0008	0.0046	0.0094	0.0146	0.0172	0.0237	0.0228	0.0276	0.0285	0.0294	0.0319
A2'	**0.0916**	**0.0591**	0.0472	0.0488	0.0462	0.0474	0.0444	0.0479	0.0454	0.0466	0.0457
	p=4										
Ml	0.0041	0.0139	0.0217	0.0271	0.0296	0.0336	0.0365	0.0371	0.0392	0.0397	0.0382
M2	**0.1425**	**0.1033**	**0.0843**	**0.0767**	**0.0699**	**0.0631**	**0.0636**	**0.062**	**0.0659**	**0.0634**	**0.0609**
MSK	0.0026	0.0114	0.0183	0.025	0.0268	0.0305	0.0328	0.0343	0.0346	0.0363	0.0359
A2	0.0001	0.004	0.0105	0.0166	0.0193	0.0195	0.0248	0.0289	0.0317	0.0312	0.0324
A2`	**0.1958**	**0.1013**	**0.0717**	**0.0633**	**0.0578**	**0.0544**	**0.0524**	**0.053**	**0.057**	**0.0519**	**0.0533**

Source: own computations.

Taking into account the critical values estimated via the Monte Carlo, the sizes of the tests M1, M2, MSK are close to the assumed significance levels independently of the sample size (see table 2). That’s why these values should be used for small samples (n<80).

Table 2. The sizes of selected tests of multivariate normality for p=2,3,4, α=0,05 i n=20,30, 40, 50, 60, 70, 80, 90,100, 110, 120 on quantile basis.

Test statistics	Sample size (n)										
	20	30	40	50	60	70	80	90	100	110	120
1	2	3	4	5	6	7	8	9	10	11	12
	p=2										
Ml	0,0495	0,0502	0,0498	0,0500	0,0500	0,0514	0,0499	0,0492	0,0494	0,0483	0,0499
M2	0,0513	0,0492	0,0500	0,0507	0,0491	0,0498	0,0498	0,0502	0,0497	0,0509	0,0506
MSK	0,0465	0,0505	0,0493	0,0496	0,0507	0,0502	0,0497	0,0486	0,0488	0,0489	0,0492
A2	0,1272	0,0523	0,0380	0,0367	0,0340	0,0323	0,0316	0,0326	0,0316	0,0342	0,0342
A2’	0,1272	0,0523	0,0380	0,0367	0,0340	0,0323	0,0316	0,0326	0,0316	0,0342	0,0342
	p=3										
Ml	0,0496	0,0506	0,0502	0,0500	0,0498	0,0478	0,0500	0,0514	0,0497	0,0490	0,0494
M2	0,0498	0,0520	0,0512	0,0504	0,0499	0,0485	0,0490	0,0504	0,0491	0,0484	0,0492
MSK	0,0500	0,0505	0,0500	0,0501	0,0503	0,0483	0,0498	0,0522	0,0498	0,0490	0,0500
A2	0,3977	0,1691	0,1031	0,0805	0,0676	0,0571	0,0555	0,0551	0,0515	0,0492	0,0503
A2’	0,3977	0,1691	0,1031	0,0805	0,0676	0,0571	0,0555	0,0551	0,0515	0,0492	0,0503
	p=4										
Ml	0,0498	0,0510	0,0508	0,0500	0,0503	0,0487	0,0501	0,0500	0,0501	0,0509	0,0517
M2	0,0496	0,0492	0,0494	0,0486	0,0500	0,0513	0,0507	0,0491	0,0471	0,0490	0,0479
MSK	0,496	0,0513	0,0505	0,0499	0,0505	0,0488	0,0499	0,0502	0,0502	0,0511	0,0513
A2	0,6898	0,3564	0,2111	0,1526	0,1264	0,1085	0,0898	0,0817	0,0776	0,0733	0,0699
A2’	0,6898	0,3664	0,2111	0,1526	0,1264	0,1085	0,0898	0,0817	0,0776	0,733	0,0699

Source: own computations.

Tests A2 and A2' with the use of the quantile based critical values, have exactly the same power for each *n* and each *p* (see table 2). Unfortunately, for higher values of alfa (alfa>0.05) under the assumptions of true hypothesis that the multivariate distribution is normal, the sizes of these tests considerably exceed the assumed levels of significance in comparison with the M1, M2, MSK tests.

References

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