A law of large numbers for the 2-dimensional Brownian semistationary process with stochastic correlation

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Abstract

In this work we prove new limit theorems for a class of Gaussian stochastic processes which does not belong to the semimartingale class: the *Brownian Semistationary process* ($\mathcal{BSS}$). The $\mathcal{BSS}$ process is used as a model for velocity fields in turbulence flows, and as a price model in markets with transaction costs. As a non-semimartingale, the existence of its quadratic variation and covariation processes does not follow from general theorems, and need to be proven explicitly in each case. In our work we prove convergence of an appropriately scaled realised covariation process and identify a limiting process, giving sufficient conditions on the covariance function of the $\mathcal{BSS}$ process to ensure convergence. The result can be viewed as an high-frequency law of large numbers, ensuring consistency of estimators for the stochastic correlation between two non-semimartingales.

Keywords: Brownian semistationary process; scaling limit theorems; high-frequency asymptotics; non-semimartingales.

1. Introduction

We work in a fixed time horizon and sample our process at a discrete set of observations, and we let the number of observations go to infinity. Thus we are in the so-called framework of *infill asymptotics*. It is well known that if X is a semimartingale, then for any sequence of partitions of $[0,t]$, where $0 = t_0 < t_1 < \dots t_n = t$, whose mesh goes to zero, the sequence of random variables:

$$\sum_{i=1}^{n} \left(X_{t_{i+1}} - X_{t_i}\right)^2$$

converges in probability to the quadratic variation of the semimartingale X accumulated up to time t, which is a.s. finite, for all t. We can actually say more, that is, for X a semimartingale, convergence holds in the stronger sense of *u.c.p.* convergence of processes.
When X lies outside the semimartingale class, these general theorems fail to hold in general. This fact is exploited for example in the standard proof that fractional Brownian motion with exponent $H \neq \frac{1}{2}$ is *not* a semimartingale, because its realised quadratic variation on a compact interval is infinity (when $H < \frac{1}{2}$), or 0 (when $H > \frac{1}{2}$), implying that if fractional Brownian was a semimartingale, it would have bounded variation paths, which is false (see Rogers (1997)).
For a particular class of non-semimartingales Gaussian processes, *the Brownian semistationary processes* ($\mathcal{BSS}$ for short), there are results in the literature that show that the realised quadratic variation process (or, more generally, multipower variations) still converges to a finite limit (see Barndorff-Nielsen and Schmiegel (2009), Barndorff-Nielsen, Corcuera, and Podolskij (2011).)
In this work we aim to prove a corresponding result for the quadratic covariation of two Brownian semistationary processes.
The Brownian semistationary process has been used in the context of turbulence modelling, as a model for the field of the velocity vectors in a turbulent flow (see Barndorff-Nielsen and Schmiegel (2009) and Corcuera, Hedevang, Pakkanen, and Podolskij (2013)).
We will be interested in its uses in financial modelling, even in absence of the semimartingale property.

In order to avoid arbitrage in frictionless financial markets, a classical result by Delbaen and Schachermayer (1994) showed that in order for a price process to have no arbitrage, or more precisely *no free lunch with vanishing risk* (NFLVR), it is necessary for the price process to be a semimartingale.

Nevertheless, more recently it has been shown that in markets with transaction costs, it is possible to ensure that no arbitrage holds even if non semimartingales are used as price processes, provided that they satisfy conditions that ensure existence of the so-called consistent price systems (see Bender, Pakkanen, and Sayit (2013) and Pakkanen (2011).)

In this work we aim to prove a result which we can call a *weak law of large numbers* for stochastic processes. This result has a probabilistic importance, because it allows to identify a class of non-semimartingales for which a definition of the quadratic covariation process is still possible, and also a practical importance, allowing us to estimate the stochastic correlation between two assets, in a non-frictionless market, in presence of transaction costs.

We start by recalling a definition:

Definition 1 (u.c.p. convergence)**.** The sequence of càdlàg processes $X^{(n)}$ is said to converge *uniformly on compacts in probability* (in short u.c.p.) to the càdlàg process X if, for all $t > 0$ and all $\varepsilon > 0$:

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{s<t}\left|X_s^{(n)} - X_s\right| > \varepsilon\right) = 0$$

Remark 1. Note that the assumption that the processes are càdlàg is sufficient to ensure that the supremum is $\mathbb{P}$-measurable.

2. Setting

The aim of this work is to prove asymptotic results for the product of the increments of a non-semimartingale, the Brownian semistationary process. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a filtered, complete probability space supporting two independent $\mathscr{F}_t$-Brownian measures $W^{(1)}, W^{(2)}$ on $\mathbb{R}$. Let $\mathcal{B}(\mathbb{R})$ denote the class of Borel subsets of $\mathbb{R}$. An $\mathscr{F}_t$-adapted Brownian measure $W\colon \Omega \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ is a Gaussian stochastic measure such that, if $A \in \mathcal{B}(\mathbb{R})$ with $\mathbb{E}[W(A^2)] < \infty$:

$$W(A) \sim N(0, \lambda(A)),$$

where λ is the Lebesgue measure. Moreover, if $A \subseteq [t, +\infty)$, then $W(A)$ is independent of $\mathscr{F}_t$. An introduction to constructing stochastic integrals against such measures can be found in Walsh (1986).

We recall the definition of the Brownian semistationary process from Barndorff-Nielsen, Corcuera, and Podolskij (2011) or Barndorff-Nielsen and Schmiegel (2009).

Definition 2. The one-dimensional Brownian semistationary process ($\mathcal{BSS}$) without drift is defined as:

$$Y_t = \int_{-\infty}^{t} g(t-s)\,\sigma_s dW_s, \tag{1}$$

where W is an $\mathscr{F}_t$-adapted Brownian measure, σ is càdlàg and $\mathscr{F}_t$-adapted, $g\colon \mathbb{R} \to \mathbb{R}$ is a deterministic function, continuous in $\mathbb{R} \setminus \{0\}$, with $g(t) = 0$ if $t \leq 0$ and $g \in L^2((0,\infty))$. We also need to impose that $\int_{-\infty}^{t} g^2(t-s)\sigma_s^2\, ds < \infty$ a.s. so that a.s. we have $Y_t < \infty$ for all $t \geq 0$.

We are interested in defining a two-dimensional Brownian semistationary process.

Take $W^{(1)}$ and $\tilde{W}$ two independent Brownian measures and consider a continuous stochastic process $(\rho_t)_{t\in\mathbb{R}}$ defined on the whole real line.

Assumption 1. *ρ has continuous sample paths, is independent of $W^{(1)}$ and $\tilde{W}$, and its paths lie in the interval $[-1, +1]$.*

Definition 3 (Two-dimensional $\mathcal{BSS}$ without stochastic volatility)**.** Consider the two processes:

$$Y_t^{(1)} := \int_{-\infty}^{t} g^{(1)}(t-s)\sigma_s^{(1)}\, dW_s^{(1)}$$

$$Y_t^{(2)} := \int_{-\infty}^{t} g^{(2)}(t-s)\sigma_s^{(2)}\rho_s\, dW_s^{(1)} + \int_{-\infty}^{t} g^{(2)}(t-s)\sigma_s^{(2)}\sqrt{1-\rho_s^2}\, d\tilde{W}_s.$$

The vector process:

$$(\mathbf{Y}_t)_{t\in\mathbb{R}} = \begin{pmatrix} Y_t^{(1)} \\ Y_t^{(2)} \end{pmatrix}_{t\in\mathbb{R}}$$

is defined to be a 2-dimensional correlated Brownian semistationary process without stochastic volatility.

We begin by first illustrating the case when $\sigma^{(1)}$ and $\sigma^{(2)}$ are constant processes equal to 1. As we will see, the core part of the proof is to show that convergence holds in this case first. The setting we consider is that of a finite, fixed time horizon $[0, T]$. We will discretely sample our processes along successive partitions of $[0, T]$. The mesh of the partition will shrink at rate $\frac{1}{n}$, and we denote the mesh at the $n - th$ step with Δ_n. We are in the setting of the so-called *infill asymptotics*.
Let $\Delta_i^n Y := Y_{i\Delta_n} - Y_{(i-1)\Delta_n}$, we compute

$$\Delta_i^n Y^{(1)} = \int_{-\infty}^{(i-1)\Delta_n} \left(g^{(1)}\left(i\Delta_n - s\right) - g^{(1)}\left((i-1)\Delta_n - s\right) \right) dW_s^{(1)} + \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s)\, dW_s^{(1)},$$
$$\Delta_i^n Y^{(2)} = \int_{-\infty}^{(i-1)\Delta_n} \left(g^{(2)}\left(i\Delta_n - s\right) - g^{(2)}\left((i-1)\Delta_n - s\right) \right) dW_s^{(2)} + \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(2)}(i\Delta_n - s)\, dW_s^{(2)}.$$

Outside the semimartingale class, there is no warranty that the limit:

$$\mathbb{P} - \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)}$$

exists. Our aim is to show that, even in absence of general theorems, we can prove convergence and identify the limiting process (subject to appropriate scaling).

3. Existence of finite limit at 0

In this Section we will be working with the following assumptions:

Assumption 2. *We assume that $g^{(i)}$, $i \in \{1, 2\}$ are nonnegative functions, belonging to $L^2((0, +\infty))$.*

Assumption 3. *We assume $g^{(i)}$, $i \in \{1, 2\}$, to be differentiable everywhere, with derivative $\left(g^{(i)}\right)' \in L^2((b, \infty))$ for some $b > 0$ and $\left((g^{(i)})'\right)^2$ non-increasing in $[b, \infty)$.*

Remark 2. Note that we are not assuming that $\left(g^{(i)}\right)' \in L^2((0, \infty))$, otherwise we would be back to the semimartingale case. In particular, we must have that, for all $\varepsilon > 0$, $\sup_{x\in(0,\varepsilon)} \left(g^{(i)}\right)'(x) = +\infty$.

Assumption 4. *The limit:*

$$g^{(1)}(0+)g^{(2)}(0+) := \lim_{x\to 0^+} g^{(1)}(x)g^{(2)}(x),$$

exists finite.

Assumption 5. *The paths of ρ are almost surely Hölder continuous with exponent α.*

Assumption 6. *We finally require that, for $i \in \{1, 2\}$, the quantities:*

$$\int_0^{\Delta_n} \left(g^{(i)}(s)\right)^2 ds \qquad \text{and} \qquad \int_0^1 \left(g^{(i)}(s + \Delta_n) - g^{(i)}(s)\right)^2 ds$$

are asymptotic to $Cx^{2\delta^{(i)}+1}$, for $x \to 0^+$, for a positive constant C, and $\delta^{(i)} \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$.

Example 1. *The gamma kernel:*

$$g(x) = e^{-\lambda x} x^{\delta} 1_{\{x>0\}},$$

is very relevant in turbulence modelling, (see Corcuera, Hedevang, Pakkanen, and Podolskij (2013)) indeed $g(x) \sim x^{\delta}$, close to 0, and δ is called the scaling parameter*, since this x^{δ} relation corresponds to Kolmogorov's scaling law in turbulence.*
Note that with $\delta \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$, g satisfies Assumptions 3 and 6 (a proof of the second statement is contained in Barndorff-Nielsen, Corcuera, and Podolskij (2011)).

With this assumptions at hand, we prove the following theorem:

Theorem 1. *Suppose that the above Assumptions hold. Then the following convergence holds:*

$$\sum_{i=1}^{\lfloor n\cdot\rfloor} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)} \overset{u.c.p.}{\to} g^{(1)}(0+)g^{(2)}(0+) \int_0^{\cdot} \rho_s \, ds. \tag{2}$$

A full detailed proof of this result can be found in Granelli and Veraart (2015). Here we only give an indication of how we can proceed to prove the result. The crucial part is to introduce the term:

$$\sum_{i=1}^{n} \mathbb{E}\left[\Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)} \middle| \mathscr{G}_{(i-1)\Delta_n} \right] \tag{3}$$

where

$$\mathscr{G}_{(i-1)\Delta_n} = \sigma\{\rho_s; s \in \mathbb{R}\} \vee \sigma\{W_u^{(1)} - W_t^{(1)}, \tilde{W}_u - \tilde{W}_t; \quad -\infty < t \leq u \leq (i-1)\Delta_n\}.$$

The proof is then done in two steps using L^2 techniques, showing first convergence of:

$$\left| \sum_{i=1}^{n} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)} - \sum_{i=1}^{n} \mathbb{E}\left[\Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)} \middle| \mathscr{G}_{(i-1)\Delta_n} \right] \right|$$

to 0, and subsequently prove that the term in (3) converges to the desired limit as stated in Theorem 1.

3. Scaling limit

The limit theorem obtained in the previous Section requires the existence of the limit at 0+ of the product $g^{(1)}(x)g^{(2)}(x)$.

For functions g for which this is not the case, a notable example being the gamma function $g(x) = x^{\delta}e^{-\lambda x}$ for $\delta < 0$, we need to follow a different approach, similar to the one laid out in Barndorff-Nielsen and Schmiegel (2009).

We need to introduce some more notation. We will write:

$$\Delta_i^n Y^{(1)} = \int_{-\infty}^{i\Delta_n} \varphi_{\Delta_n}^{(1)}(i\Delta_n - s) \, dW_s^{(1)},$$

where

$$\varphi_{\Delta_n}^{(1)}(i\Delta_n - s) = \begin{cases} g^{(1)}(i\Delta_n - s) & s \geq (i-1)\Delta_n \\ g^{(1)}(i\Delta_n - s) - g^{(1)}((i-1)\Delta_n - s) & s < (i-1)\Delta_n \end{cases},$$

and similarly for $\Delta_i^n Y^{(2)}$.

Assumption 7. *We assume that $g^{(1)}(x), g^{(2)}(x)$ are both decreasing on $\mathbb{R}^+$.*

We obtain the following result:

Theorem 2. *In presence of the above assumptions, we obtain:*

$$\Delta_n \frac{\sum_{i=1}^{\lfloor n\cdot\rfloor} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)}}{c(\Delta_n)} \overset{u.c.p.}{\to} \int_0^{\cdot} \rho_l \, dl,$$

where the scaling factor is defined to be:

$$c(\Delta_n) := \int_0^{\Delta_n} g^{(1)}(s)g^{(2)}(s) \, ds + \int_0^{+\infty} \left(g^{(1)}(s+\Delta_n) - g^{(1)}(s)\right) \left(g^{(2)}(s+\Delta_n) - g^{(2)}(s)\right) ds$$

We note that this result differs from the result obtained in Theorem 1 by the fact that we need to include the additional scaling factor $\frac{\Delta_n}{c(\Delta_n)}$.
The proof for this result is again done in two parts, but is very different from the case where the limit at 0 exists. The proof is based on showing that, under our assumptions, a weak convergence of suitable probability measures holds, allowing us to identify the limiting process. Subsequently, one needs to ensure that the covariance structures of the processes allow the conditional variance of the realised covariation process to asymptotically approach 0. We again refer to Granelli and Veraart (2015) for the full details of the proof.

4. General results with volatility

In this Section we finally discuss the forms that our theorems assume when the stochastic volatility is added. Here we assume that $\sigma^{(1)}, \sigma^{(2)}$ are càdlàg stochastic processes mutually independent and independent of $W^{(1)}, \tilde{W}, \rho$.
Define:

$$Y_t^{(1)} = \int_{-\infty}^{t} g^{(1)}(t-s)\sigma_s^{(1)}\, dW_s^{(1)},$$

and

$$Y_t^{(2)} = \int_{-\infty}^{t} g^{(2)}(t-s)\sigma_s^{(2)}\, dW_s^{(2)}.$$

We obtain:

$$\Delta_i^n Y^{(1)} = \int_{-\infty}^{(i-1)\Delta_n} \left(g^{(1)}(i\Delta_n - s) - g^{(1)}(i-1)\Delta_n - s)\right)\sigma_s^{(1)}\, dW_s^{(1)} + \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(1)}(i\Delta_n - s)\sigma_s^{(1)}\, dW_s^{(1)},$$
$$\Delta_i^n Y^{(2)} = \int_{-\infty}^{(i-1)\Delta_n} \left(g^{(2)}(i\Delta_n - s) - g^{(2)}(i-1)\Delta_n - s)\right)\sigma_s^{(2)}\, dW_s^{(2)} + \int_{(i-1)\Delta_n}^{i\Delta_n} g^{(2)}(i\Delta_n - s)\sigma_s^{(2)}\, dW_s^{(2)}.$$

We will need the following technical assumption:

Assumption 8. *We assume that for $i \in \{1, 2\}$, we have:*

$$\int_1^{\infty} \left(\frac{d}{ds} g^{(i)}(s)\right)^2 \sigma_{y-s}^{(i)}\, ds < \infty, \qquad \textit{for all } y \in \mathbb{R}^+.$$

With this final assumption in place, our result takes the following form:

Theorem 3. *For the normalised quadratic covariation we have the following convergence:*

$$\Delta_n \frac{\sum_{n=1}^{\lfloor n\cdot \rfloor} \Delta_i^n Y^{(1)} \Delta_i^n Y^{(2)}}{c(\Delta_n)} \xrightarrow{u.c.p.} \int_0^{\cdot} \sigma_l^{(1)} \sigma_l^{(2)} \rho_l\, dl.$$

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