## Functional linear models driven by point processes

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### Abstract

We introduce a new class of functional models where a point process is part of the model and plays a key role in it. We focus on a functional linear model where the kernel is a stochastic point process, $\mathbf{N}$. If $\mathbf{N}$ is a Poisson process, we derive unbiased and consistent estimators of the intensity of $\mathbf{N}$, in the homogeneous case, and of the expectation measure of $\mathbf{N}$, in the non homogeneous case. We observe that confidence intervals and bands for these quantities can also determined. Asymptotic normality of these estimators is obtained under regularity conditions on the functional noise.

**Keywords**: Poisson process; Functional data analysis; Estimation of intensity; Expectation measure.

### 1. Introduction

Functional linear models have been extensively studied and there is large literature about them. The most common forms of functional linear models are those that admit the integral representations given by

$$\mathcal{Y} = \int_0^T \beta(s)\mathcal{X}(s)ds + \mathcal{E} \tag{1}$$

and

$$\mathcal{Y}(t) = \int_0^T \beta(s,t)\mathcal{X}(s)ds + \mathcal{E}(t) \tag{2}$$

with scalar and functional responses, respectively. Here, the functional parameter $\beta(s)$, in the scalar response case, or $\beta(s,t)$, in the functional response case, is deterministic and often is the object to be estimated, based on a sample of responses $\{\mathcal{Y}_i : 1 \leq i \leq N\}$ and functional covariates $\{\mathcal{X}_i : 1 \leq i \leq N\}$ . See Dabo-Niang and Ferraty (2008), Ferraty (2011), Ferraty and Romain (2011), Ferraty and Vieu (2006), Horváth and Kokoszka (2012), Ramsay and Silverman (2005) and references therein for a comprehensive exposition of functional data analysis.

One possible way of generalization of these models is to consider models where these $\beta$'s are stochastic processes. In this work we will study the case where the kernel $\beta(s,t)$ is replaced by a stochastic *point* process. More precisely, we will be interested in the following model:

$$\mathcal{Y}(t) = \int_0^t \mathbf{N}((s,t])\mathcal{X}(s)ds + \mathcal{E}(t) \tag{3}$$

where $\mathbf{N}$ is a point process on $\mathbb{R}_+$. We assume that both $\mathcal{Y}$ and $\mathcal{X}$ are defined on $[0,T] \subset \mathbb{R}$, and that $\mathcal{X}(t)$ is strictly positive and integrable on $[0,T]$.

Furthermore, in this work we will assume that $\mathbf{N}$ is a Poisson process with no fixed atoms, which can be either homogeneous or non homogeneous. In case $\mathbf{N}$ is homogeneous, we will estimate its intensity, $\lambda$, and if it is non homogeneous, we will estimate its expectation measure, through the cadlag function $\Lambda$, where $\mathbb{E}\mathbf{N}((a,b]) = \Lambda(b) - \Lambda(a)$, and $\Lambda(0) = 0$. See Brillinger (1975), Daley and Vere-Jones (1988), de Miranda and Morettin (2005, 2006 and 2011), Karr (1986) and references therein for point processes and their estimation.

**2. The estimators of $\lambda$ and $\Lambda(t)$**

Let $\{(\mathcal{X}_i, \mathcal{Y}_i) : 1 \leq i \leq N\}$ be the set of observations from

$$\mathcal{Y}_i(t) = \int_0^t \mathbf{N}_i((s,t])\mathcal{X}_i(s)ds + \mathcal{E}_i(t) \tag{4}$$

where, $\mathbf{N}_i \sim \mathbf{N}$ and $\mathcal{E}_i \sim \mathcal{E}$, for all $1 \leq i \leq N$ and $\{\mathbf{N}_i, \mathcal{E}_i : 1 \leq i \leq N\}$ is an independent set of random elements. The following are the estimators of the intensity and the cumulative intensity of $\mathbf{N}$. We remind that these quantities determine the law of the Poisson process $\mathbf{N}$.

**The homogeneous case:**

$$\hat{\lambda} := \frac{\sum_{i=1}^N \int_0^T \mathcal{Y}_i(t)\mathrm{d}t}{\sum_{i=1}^N \int_0^T \int_0^t (t-s)\mathcal{X}_i(s)\mathrm{d}s\mathrm{d}t} \tag{5}$$

**The non homogeneous case:**

$$\hat{\Lambda}(t) := \frac{\sum_{i=1}^N \mathcal{Y}_i(t)}{\sum_{i=1}^N \int_0^t \mathcal{X}_i(s)\mathrm{d}s} \tag{6}$$

**3. Main results**

The estimators (5) and (6) have good properties. We have the following theorems:

**Theorem 1:** *Let, for all $t \in (0,T]$, $\texttt{lim inf}\frac{1}{N}\int_0^t \sum_{i=1}^N \mathcal{X}_i(s)\mathrm{d}s > 0$. Assume that in model (4) we have $\mathcal{E}$ such that for all $t \in [0,T]$, $\mathbb{E}\mathcal{E}(t) = 0$ and $\mathbb{V}\texttt{ar}\mathcal{E}(t)$ is finite. Suppose that $\mathbf{N}$ is an homogeneous Poisson process. Then $\hat{\lambda}$ is an unbiased estimator of $\lambda$ such that $\mathbb{V}\texttt{ar}(\hat{\lambda}) \to 0$ as $N \to \infty$. Moreover, there exists a constant $c \in \mathbb{R}_+$ such that $\sqrt{N}(\hat{\lambda} - \lambda) \rightsquigarrow \mathcal{N}(0,c)$ as $N \to \infty$.*

**Theorem 2:** *Let $(\mathcal{X}_i)_{i\in\mathbb{N}^*}$ be as in Theorem 1. Assume that in model (4) we have $\mathcal{E}$ such that for all $t \in [0,T]$, $\mathbb{E}\mathcal{E}(t) = 0$ and $\mathbb{V}\texttt{ar}\mathcal{E}(t)$ is finite. Suppose that $\mathbf{N}$ is a non homogeneous Poisson process. Then $\hat{\Lambda}$ is an unbiased estimator of $\Lambda$ such that $\mathbb{V}\texttt{ar}(\hat{\Lambda}) \to 0$ as $N \to \infty$. Moreover, if we assume that $\mathcal{E}$ is a Gaussian process or let it have independent increments, then $\sqrt{N}(\hat{\Lambda} - \Lambda)$ converges in distribution to a zero mean Gaussian process on $[0,T]$ as $N \to \infty$.*

**4. Conclusion**

Although we have defined and studied here a functional linear model driven by a Poisson point process, a wide class of new functional models can be constructed based on the replacement of the functional parameters in a general, not necessarily linear, functional model by stochastic point processes, which, by their turn, can also be quite general point processes with internal dependence.

We observe that, although we have unbiasedness and good asymptotic properties for $\hat{\Lambda}$, in practice, we expect to have wild amplitudes and variations of $\hat{\Lambda}(t)$ for sufficiently small values of $t$. This is due to the fact that, for a fixed finite $N$, we have $\lim_{t\to 0}\sum_{i=1}^N \int_0^t \mathcal{X}_i(s)\mathrm{d}s = 0$ and $\lim_{\delta\to 0}\mathbb{P}(\forall i, 1 \leq i \leq N, \mathbf{N}_i([0,\delta)) = 0) = 1$, and, consequently, we will have, with probability that approaches one, for sufficiently small $t$ that $\mathcal{Y}_i(t) = \mathcal{E}_i(t)$ for all $i$, $1 \leq i \leq N$. Thus, assuming that $\mathcal{E}$ is such that for any fixed $t$ we have $\mathbb{P}(\mathcal{E}(t) = 0) = 0$, which in many practical situations is a sensible assumption, we will have $\hat{\Lambda}(t) = \frac{\sum_{i=1}^N \mathcal{E}_i(t)}{\sum_{i=1}^N \int_0^t \mathcal{X}_i(s)\mathrm{d}s}$ and either the limit $\lim_{t\to 0}\hat{\Lambda}(t)$ does not exist or $\lim_{t\to 0}\hat{\Lambda}(t) = \pm\infty$ with probability that approaches one as $t$ goes to zero. This will depend, for example, on the continuity of the noise process. A way out of this situation, is not to take into account the estimates for small values of $t$ and consider only those values of the estimates where we observe a steady increasing or at least non decreasing of their values as a function of $t$, which in addition must be all non negative ones. The reason for this is that $\Lambda$ is by definition a non decreasing function of $t$ that takes only non negative values. Smoothing of $\hat{\Lambda}$ is also a possibility to improve the estimate. Finally, we observe that one can also obtain confidence intervals and bands for these estimators.

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