Quadratic forms on complex elliptical random variables and its applications

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Abstract

Quadratic forms on complex random elliptical matrices and their joint eigenvalue densities are derived, and these densities are represented by complex hypergeometric functions of matrix arguments which can be expressed in terms of complex zonal polynomials. An integral representation of this quadratic form is introduced. The connection between these densities and information theory is discussed. Special cases are described and select applications highlighted.

Keywords: eigenvalues; information theory; matrix variate complex elliptical distribution; Wishart distribution.

1. Introduction

The matrix variate complex elliptical distribution can be expressed as an integral of a series of complex normal densities, having the same covariance structure to different scales (Arashi (2013)). Therefore, properties of the matrix variate complex elliptical distribution can be easily explored due to the fact that it can be described in terms of its complex normal components. The different choices for the weight functions for the random variable emphasize the flexibility of this representation of the matrix variate complex elliptical distribution. In this paper, a generalised complex distribution in terms of integral series of complex quadratic densities is proposed, with the complex Wishart density as a special case. Special focus is given to the densities of the eigenvalues (joint and maximum) of this distribution and the application thereof in information theory.

2. Integral series of complex quadratic densities distribution

The complex random matrix $\boldsymbol{X}: n \times p$ has a matrix variate complex elliptical distribution if its characteristic function has the form

$$\phi_{\boldsymbol{X}}(\boldsymbol{T}) = exp\left(i\mathrm{tr}(\boldsymbol{T}^H\boldsymbol{M})\right)\psi\left(\mathrm{tr}(\boldsymbol{\Sigma_1 T \Sigma_2 T}^H)\right), \qquad (1)$$

where $\boldsymbol{T}, \boldsymbol{M} \in M_{n\times p}(\mathbb{C}), \boldsymbol{\Sigma_1} = \boldsymbol{CC}^H \in M_p(\mathbb{C}), \boldsymbol{\Sigma_2} = \boldsymbol{BB}^H \in M_n(\mathbb{C})$ are both invertible, i the usual imaginary complex unit and $\psi(.): [0,\infty) \to \mathbb{R}$ is the characteristic generator. The density function of $\boldsymbol{X}$ can be expressed as

$$f_{\boldsymbol{X}}(\boldsymbol{X}) = g\left[-\mathrm{tr}\left(\boldsymbol{\Sigma_1}^{-1}(\boldsymbol{X}-\boldsymbol{M})\boldsymbol{\Sigma_2}^{-1}(\boldsymbol{X}-\boldsymbol{M})^H\right)\right]$$

where $g(.)$ is the density generator satisfying the conditions under the real case.
The quadratic form on $\boldsymbol{X}$ associated with the positive definite Hermitian matrix $\boldsymbol{A}$ is defined by
$\boldsymbol{S} = \boldsymbol{X}^H\boldsymbol{AX}$.
We focus specifically on the distribution of $\boldsymbol{S}$.

It is demonstrated by Arashi et. al. (2014) that (1) can be expanded as an integral series of complex matrix variate normal densities. This forms the platform for the new proposed integral series of complex quadratic densities distribution with the complex Wishart as a special case i.e. where $\boldsymbol{A} = \boldsymbol{I}_p, \boldsymbol{\Sigma_1} = \boldsymbol{I}_n,$ and $\boldsymbol{\Sigma_2} = \boldsymbol{\Sigma}$. Attention is given to the joint density of the ordered eigenvalues as well as the uncorrelated case i.e. $\boldsymbol{\Sigma_2} = \sigma^2 \boldsymbol{I}_p$.

The main result derived in this paper proposes an alternative method for translating some of these cumbersome mathematical expressions into convenient form, making these methods more practically tractable.

3. Application

An application, namely the Wilks' statistic, arises in the context of hypothesis testing of a mean matrix, the reader is referred to the work of Loots et. al. (2011) and Arashi et. al. (2014), and is briefly discussed. With specific choice of the weight function, the applicability of this new model within the information theory context is also discussed.

4. Conclusions

An integral representation for the density function of the matrix variate complex elliptical distribution is derived and studied. This representation eases theoretical derivations of the properties of the matrix variate complex elliptical distribution, as well as the development of the generalised complex quadratic distribution in terms of integral series of complex quadratic densities. Applications including Wilks' statistic is discussed including connections to applicability in information theory.

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