Transmuted generalized Gompertz distribution for modelling reliability data

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Abstract

This paper introduces the four parameter transmuted generalized Gompertz distribution which is a generalization of the generalized Gompertz distribution and studies its statistical properties. The proposed model possesses the increasing or decreasing bathtub shape for its instantaneous failure rates. We obtain the analytical shapes of density and hazard functions. Explicit expressions are derived for the quantile, moments and entropy. The model parameters are estimated by the method of maximum likelihood. Finally, an application of the new distribution is illustrated using reliability data.

Keywords: Reliability functions; moment estimation; maximum likelihood estimation.

1. Introduction

The Gompertz distribution was pioneered to modeling human mortality data by Gompertz (1825). The Gompertz distribution has wide applications in demography, actuarial and biological sciences. The maximum likelihood estimates for the parameters of the Gompertz distributional was studied by Garg et al. (1970). Recently Gohary et al. (2011) proposed the generalized Gompertz distribution and compare this model with exponential family of lifetime distributions, which has the following cdf as

$$G(x;\alpha,\beta,\eta) = \left[1-\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\right]^{\beta}, \qquad (x>0), \tag{1}$$

The probability density function corresponding to (1) is given by

$$g(x;\alpha,\beta,\eta) = \alpha\beta e^{\eta x}\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\left[1-\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\right]^{\beta}, \tag{2}$$

respectively. where $\alpha,\eta > 0$ are the scale parameters and β is the shape parameter.

Recently sevaral distributions have been proposed from the transmuted family of lifetime distributions. Aryal et al. (2011) proposed the transmuted Weibull distributions with applications. Merovci (2013) studied the transmuted Rayleigh distribution with an application to bladder cancer data. More recently Khan and King (2014a) proposed the transmuted Inverse Weibull distribution and discussed theoretical properties of this distribution with application to lifetime data. Khan et al. (2014b) proposed the transmuted Gompertz distribution and studied various structural properties with an application to reliability data. Yuzhu et al. (2014) proposed the transmuted linear exponential distribution and compare its goodness of fit with the twelve different lifetime distributions. The motivation of this study is to introduce new model which has the capability to describe bathtub shaped failure rate function.

By using the quadratic rank transmutation map technique proposed by Shaw et al. (2007), we develop the four parameter transmuted generalized Gompertz distribution. According to this approach a random variable X is said to have a transmuted distribution if its cumulative distribution function (cdf) satisfies the following relationship

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2, \qquad |\lambda| \le 1 \tag{3}$$

and

$$f(x) = g(x)\{(1+\lambda) - 2\lambda G(x)\}, \tag{4}$$

where $G(x)$ is the cdf of the base distribution, $g(x)$ and $f(x)$ are the corresponding probability density functions (pdf) associated with $G(x)$ and $F(x)$, respectively. It is important to note that at $\lambda = 0$ we have the distribution of the base random variable. The paper is organised as follows, in Section 2, we present the analytical shapes of the probability density and hazard function of the four parameter transmuted generalized Gompertz distribution. We formulated the moments and Rényi entropy in Section 3. Maximum likelihood estimates (MLEs) of the unknown parameters are discussed in Section 4. The usefulness of the proposed model is illustrated in Section 5. Concluding remarks are addressed in Section 6.

2. Transmuted generalized Gompertz distribution

The cumulative distribution function (CDF) of the TGG can be obtained from (3) as

$$F(x;\alpha,\beta,\eta,\lambda) = \left[1 - \exp\left\{-\frac{\alpha}{\eta}(e^{\eta x} - 1)\right\}\right]^{\beta} \mathcal{U}_1, \tag{5}$$

$$\mathcal{U}_g = \left\{1 + \lambda - g\lambda\left[1 - \exp\left\{-\frac{\alpha}{\eta}(e^{\eta x} - 1)\right\}\right]^{\beta}\right\}, \qquad g = 1,2. \tag{6}$$

where $\alpha,\beta,\eta > 0$ and $\lambda \le 1$. A random variable X having the transmuted generalized Gompertz distribution and denoted as $X \sim TGG(x;\alpha,\beta,\eta,\lambda)$. The pdf of X is given by

$$f(x;\alpha,\beta,\eta,\lambda) = \alpha\beta e^{\eta x}\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x} - 1)\right\}\left[1 - \exp\left\{-\frac{\alpha}{\eta}(e^{\eta x} - 1)\right\}\right]^{\beta-1}\mathcal{U}_2, \quad x > 0. \tag{7}$$

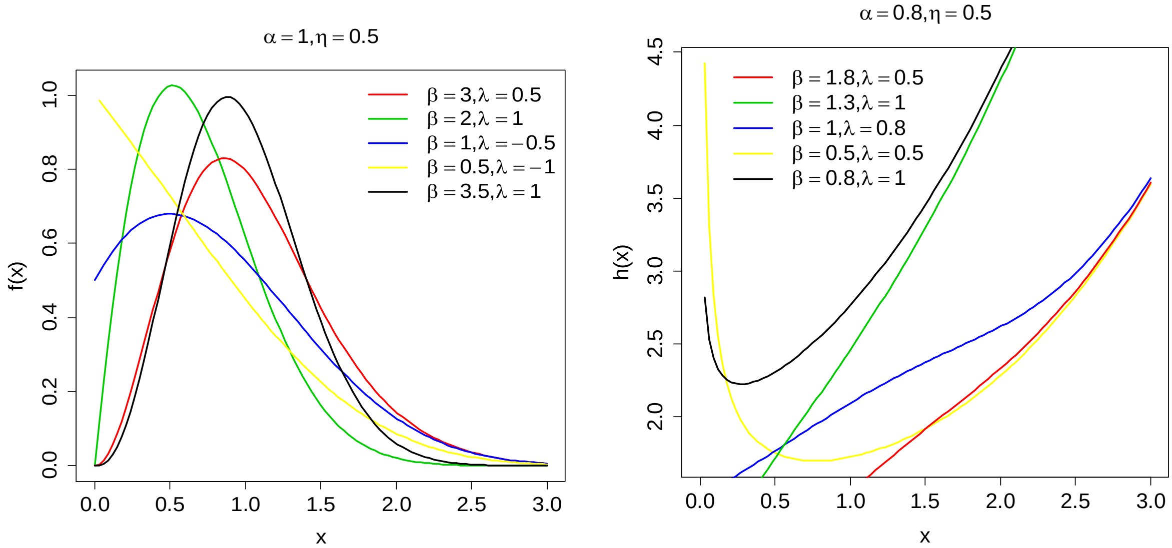

Figure: 1 plots of the TGG density and hazard functions.

The quantile function corresponding to (7) is given by

$$Q(u) = \frac{1}{\eta} ln\left[1 + ln\left\{1 - \frac{\eta}{\alpha} ln\left(1 - \left(\frac{(1+\lambda) - \sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}\right)^{\frac{1}{\beta}}\right)\right\}\right], \tag{8}$$

respectively. The hazard function is given by

$$h(x;\alpha,\beta,\eta,\lambda) = \frac{\alpha\beta e^{\eta x}\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x} - 1)\right\}\left[1 - \exp\left\{-\frac{\alpha}{\eta}(e^{\eta x} - 1)\right\}\right]^{\beta-1}\mathcal{U}_2}{1 - \left[1 - \exp\left\{-\frac{\alpha}{\eta}(e^{\eta x} - 1)\right\}\right]^{\beta}\mathcal{U}_1}, \tag{9}$$

Some possible shapes of the probability density function and hazard function are plotted in Figure 1. Figure 1 suggests that the proposed model has increasing and decreasing hazard functions.

3. Moments and Entropy

This section address the k^{th} moment and visualize the skewness and kurtosis measures. We also derive the Rényi entropy of the transmuted generalized Gompertz distribution.

Theorem 1: If X has the $TGG(x;\alpha,\beta,\eta,\lambda)$ distribution with $|\lambda|\le 1$, then the k^{th} moment of X say $\acute{\mu}_k$ is given as follows

$$\acute{\mu}_k = \left\{(1+\lambda)\sum_{i,j,k=0}^{\infty} \mathcal{V}_{i,j,k}\binom{\beta-1}{i} - 2\lambda \sum_{i,j,k=0}^{\infty} \mathcal{V}_{i,j,k}\binom{2\beta-1}{i}\right\}\Gamma(k+1).$$

Proof: By definition the k^{th} moment of X of the TGG distribution as follows

$$\acute{\mu}_k = (1+\lambda)\int_0^{\infty} x^k \alpha\beta e^{\eta x}\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\left[1-\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\right]^{\beta-1} dx -$$
$$-2\lambda\int_0^{\infty} x^k \alpha\beta e^{\eta x}\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\left[1-\exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\right]^{2\beta-1} dx,$$

Using the series expansion, we obtain

$$\acute{\mu}_k = (1+\lambda)\sum_{i,j=0}^{\infty}\frac{\alpha^{j+1}\beta(-1)^{i+j}(i+1)^j}{\eta^j\, exp\{-\alpha(i+1)/\eta\}}\binom{\beta-1}{i}\int_0^{\infty} x^k\, exp\{\eta x(j+1)\}dx -$$
$$-2\lambda\sum_{i,j=0}^{\infty}\frac{\alpha^{j+1}\beta(-1)^{i+j}(i+1)^j}{\eta^j\, exp\{-\alpha(i+1)/\eta\}}\binom{2\beta-1}{i}\int_0^{\infty} x^k\, exp\{\eta x(j+1)\}dx.$$

The above integral reduces to the k^{th} moment as

$$\acute{\mu}_k = (1+\lambda)\sum_{i,j,k=0}^{\infty}\mathcal{V}_{i,j,k}\binom{\beta-1}{i}\Gamma(k+1) - 2\lambda\sum_{i,j,k=0}^{\infty}\mathcal{V}_{i,j,k}\binom{2\beta-1}{i}\Gamma(k+1), \quad (10)$$

where

$$\mathcal{V}_{i,j,k} = \frac{\alpha^{j+1}\beta(-1)^{i+j+k+1}(i+1)^j}{\eta^j\, exp\{-\alpha(i+1)/\eta\}\{(j+1)\eta\}^{k+1}}.$$

The important features of the TGG distribution can be analysed using the k^{th} moments such as measure of central tendency, measure of dispersions, skewness and kurtosis measures. Figure 2 shows the behavior of skewness and kurtosis measures of the TGG distribution using equation (8).
The entropy of a random variable X with density $f(x)$ is a measure of variation of the uncertainty. A large value of entropy indicates the greater uncertainty in the data. The Rényi entropy defined as

$$I_R(\rho) = \frac{1}{1-\rho}log\left\{\int f(x)^{\rho}\, dx\right\},$$

where $\rho>0$ and $\rho\neq1$. The integral in $I_R(\rho)$ of the $TGG(x;\alpha,\beta,\eta,\lambda)$ can be defined as

$$\int_0^{\infty} f(x)^{\rho}dx = \int_0^{\infty}(\alpha\beta)^{\rho}e^{\rho\eta x}\, exp\left\{-\frac{\alpha\rho}{\eta}(e^{\eta x}-1)\right\}\left[1-exp\left\{-\frac{\alpha}{\eta}(e^{\eta x}-1)\right\}\right]^{\rho(\beta-1)}\mathcal{U}_2^{\rho}\, dx,$$
$$= \sum_{i,j=0}^{\infty}\mathcal{T}_{i,j}\int_0^{\infty} exp(\rho\eta x)exp\{-\alpha(\rho+j)(exp(\eta x)-1)/\eta\}\, dx, \quad (11)$$

where

$$\mathcal{T}_{i,j} = (\alpha\beta)^{\rho}\binom{\rho}{i}\binom{\rho(\beta-1)+\beta i}{j}(-1)^{i+j}\left(\frac{2\lambda}{1+\lambda}\right)^{i}(1+\lambda)^{\rho},$$

$$\int_0^\infty f(x)^\rho dx = \sum_{i,j,k=0}^{\infty} \mathcal{T}_{i,j}(-1)^k exp\{\alpha(\rho + j)/\eta\} \frac{(\rho + j)^k \alpha^k}{(\rho + k)\eta^{k+1}}.$$

Therefore, the Rényi entropy of the $TGG(x; \alpha, \beta, \eta, \lambda)$ distribution can be expressed by

$$I_R(\rho) = \frac{\rho}{1-\rho} log(\alpha) + \frac{\rho}{1-\rho} log(\beta) + \frac{\rho}{1-\rho} log(1+\lambda) + \frac{1}{1-\rho} log \left\{ \sum_{i,j,k=0}^{\infty} \binom{\rho}{i} \binom{\rho(\beta-1) + \beta i}{j} (-1)^{i+j+k} \left(\frac{2\lambda}{1+\lambda}\right)^i exp\{\alpha(\rho + j)/\eta\} \frac{(\rho + j)^k \alpha^k}{(\rho + k)\eta^{k+1}} \right\}. \quad (12)$$

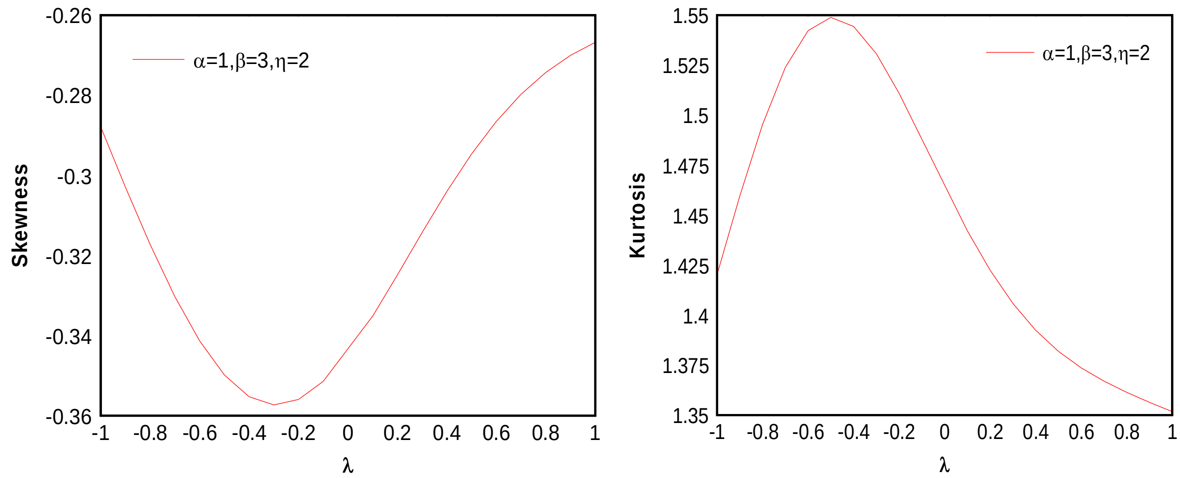

Figure: 2 Skewness and kurtosis of the TGG distribution

4. Maximum Likelihood Estimation

Consider the random samples $x_1, x_2, \ldots, x_n$ consisting of n observations from the $TGG(x; \alpha, \beta, \eta, \lambda)$ distribution having the probability density function given in equation (7). Let $\Theta = (\alpha, \beta, \eta, \lambda)^{\mathrm{T}}$ be the vector of the parameters. The log-likelihood function for Θ can be expressed as

$$\mathcal{L}(\Theta) = nln\alpha + nln\beta + \eta \sum_{i=1}^{n} x_i - \frac{\alpha}{\eta} \sum_{i=1}^{n} (e^{\eta x_i} - 1) + (\beta - 1) \sum_{i=1}^{n} ln\left[1 - exp\left\{-\frac{\alpha}{\eta}(e^{\eta x_i} - 1)\right\}\right] + \sum_{i=1}^{n} ln\left\{1 + \lambda - 2\lambda\left[1 - exp\left\{-\frac{\alpha}{\eta}(e^{\eta x_i} - 1)\right\}\right]^\beta\right\}, \quad (13)$$

By differentiating (13) with respect to α, β, η and λ then equating it to zero, we obtain the component of the score vector $U(\Theta)$ are given by

$$\frac{\partial \mathcal{L}(\Theta)}{\partial \alpha} = \frac{n}{\alpha} - \frac{1}{\eta} \sum_{i=1}^{n} (e^{\eta x_i} - 1) + (\beta - 1) \sum_{i=1}^{n} \frac{h(x_i, \alpha, \eta)(e^{\eta x_i} - 1)}{\eta[1 - h(x_i, \alpha, \eta)]} + \frac{2\lambda\beta}{\eta} \sum_{i=1}^{n} \frac{[1 - h(x_i, \alpha, \eta)]^{\beta-1} h(x_i, \alpha, \eta)(e^{\eta x_i} - 1)}{\{1 + \lambda - 2\lambda[1 - h(x_i, \alpha, \eta)]^\beta\}},$$

$$\frac{\partial \mathcal{L}(\theta)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} ln[1 - h(x_i, \alpha, \eta)] - 2\lambda \sum_{i=1}^{n} \frac{[1 - h(x_i, \alpha, \eta)]^{\beta-1} ln[1 - h(x_i, \alpha, \eta)]}{\{1 + \lambda - 2\lambda[1 - h(x_i, \alpha, \eta)]^\beta\}},$$

$$\frac{\partial\mathcal{L}(\Theta)}{\partial\eta} = \sum_{i=1}^{n} x_i - \frac{\alpha}{\eta^2}\sum_{i=1}^{n}\{e^{\eta x_i}(\eta x_i - 1) + 1\} + \frac{\alpha(\beta-1)}{\eta^2}\sum_{i=1}^{n}\frac{h(x_i,\alpha,\eta)\{e^{\eta x_i}(\eta x_i - 1) + 1\}}{1 - h(x_i,\alpha,\eta)} - \frac{2\lambda\alpha\beta}{\eta^2}\sum_{i=1}^{n}\frac{[1 - h(x_i,\alpha,\eta)]^{\beta-1}h(x_i,\alpha,\eta)\{e^{\eta x_i}(\eta x_i - 1) + 1\}}{\{1 + \lambda - 2\lambda[1 - h(x_i,\alpha,\eta)]^{\beta}\}},$$

$$\frac{\partial\mathcal{L}(\Theta)}{\partial\lambda} = \sum_{i=1}^{n}\frac{1 - 2[1 - h(x_i,\alpha,\eta)]^{\beta}}{\{1 + \lambda - 2\lambda[1 - h(x_i,\alpha,\eta)]^{\beta}\}},$$

where $h(x_i,\alpha,\eta) = exp\left\{-\frac{\alpha}{\eta}(e^{\eta x_i} - 1)\right\}$ respectively, the asymptotic variance covariance matrix of MLEs for the parameter vector $\Theta = (\alpha,\beta,\eta,\lambda)^{T}$ can be considered as the multivariate normal with the variance covariance matrix and its inverse of the expected information matrix are given by

$$\left((\hat{\alpha}-\alpha),(\hat{\beta}-\beta),(\hat{\eta}-\eta),(\hat{\lambda}-\lambda)\right) \sim N_4\{0, K(\Theta)^{-1}\},$$

where $K(\Theta)^{-1}$ is the variance covariance matrix of the unknown parameters. The multivariate normal distribution can be used to obtain an approximate $100(1-\gamma)\%$ confidence intervals for the parameters α,β,η and λ can be determined as

$$\hat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{11}}, \quad \hat{\beta} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{22}}, \quad \hat{\eta} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{22}}, \quad \hat{\lambda} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{33}},$$

Where $Z_{\frac{\gamma}{2}}$ is the upper αth percentile of the standard normal distribution.

5. Application

This section illustrates the applicability of the TGG distribution using tensile strength of 100 observations of carbon fibers data. The data have been obtained from Nicholas and Padgett (2006). To illustrate the usefulness of the transmuted generalized Gompertz (TGG) distribution for the breaking stress of carbon fibers data, we compare its goodness of fit with sub-models, namely the generalized Gompertz (GG), transmuted Gompertz (TG) and Gompertz (G) distributions. Maximum likelihood estimation was implemented by using BFGS method to minimize the log likelihood function in R. The MLEs of the TGG, GG, TG and G distributions with their corresponding standard errors, Kolmogorov-Smirnov (K-S) test, Cramér-von Mises and Anderson-Darling goodness of-fit statistics measures are displayed in Table 1. From Table 1, the smallest the values of these statistics indicate the better fit, which implies that the TGG distribution provides the better fit than the other three distributions. Plots of the fitted densities and cdfs for the TGG, GG, TG and G distributions for the carbon fibers data are displayed in Figure 3.

Table 1: MLEs of the unknown Parameters for carbon fibers data with their corresponding SE in parenthesis, KS-Test, Cramér-von Mises and Anderson-Darling goodness of-fit statistics

Distribution	Parameter Estimates				K-S test	$\mathcal{W}$	$\mathcal{A}$
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\lambda}$			
TGG	0.2689	2.9105	0.3949	0.6816	0.0612	0.0701	0.4107
	(0.1174)	(0.8954)	(0.1451)	(0.3462)			
GG	0.4230	3.4520	0.3360	-	0.0641	0.0789	0.4616
	(0.1486)	(1.0933)	(0.1160)				
TG	0.0381	-	0.8996	0.7354	0.0860	0.1400	1.1239
	(0.0097)		(0.0842)	(0.1718)			
G	0.0781	-	0.7886	-	0.1025	0.1768	1.3637
	(0.0176)		(0.0773)				

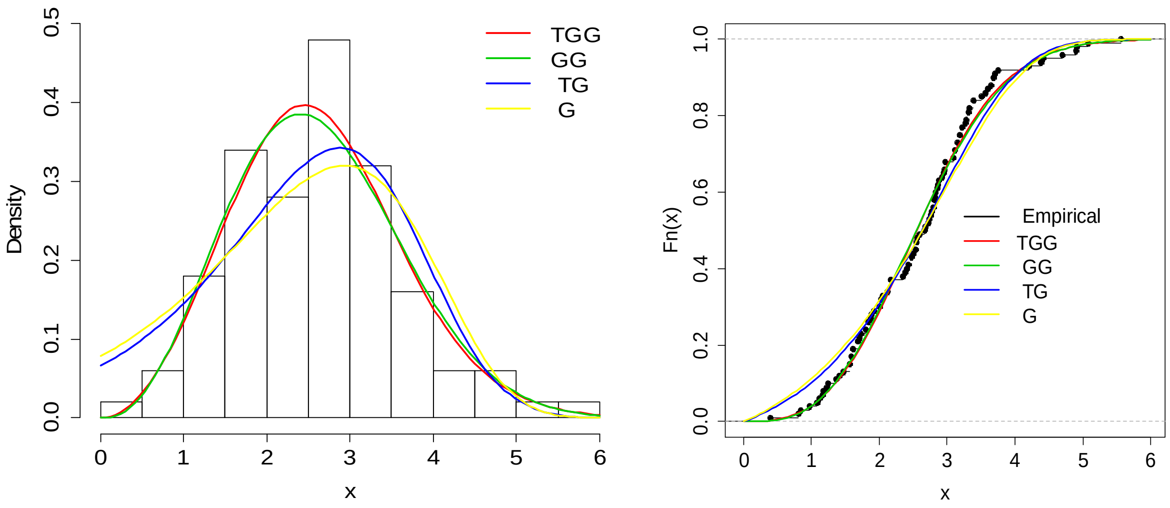

Figure: 3 Plots of the estimated densities and cdfs of the TGG, GG, TG and G distributions

6. Conclusions

In this research we propose the four parameter transmuted generalized Gompertz distribution. Some mathematical treatment of the explicit expressions are derived such as density and hazard functions, quantile, moments and Rényi entropy. Maximum likelihood estimation is proposed for estimating the unknown parameters. Finally a real data set was analysed to show the potential of the TGG model.

References

1. Aryal Gokarna R., Tsokos Chris P. (2011) Transmuted Weibull distribution: A Generalization of the Weibull Probability Distribution. European Journal of Pure and Applied Mathematics, Vol. 4: 2, 89-102.
2. Gompertz, B. (1825). "On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingencies". Philosophical Transactions of the Royal Society of London 115: 513–583.
3. Garg M.L., Rao B.R., Redmond C.K., (1970). Maximum likelihood estimation of the parameters of the Gompertz survival function, Journal of the Royal Statistical Society C 19, 152–159.
4. Gohary A. El, Alshamrani Ahmad, Otaibi .Adel Naif Al, (2013). The generalized Gompertz distribution, Applied Mathematical Modelling, 37: 13–24.
5. Khan M. Shuaib and King Robert. (2014a). A New Class of Transmuted Inverse Weibull Distribution for Reliability Analysis, American Journal of Mathematical and Management Sciences, Vol. 33:4, 261-286.
6. Khan, M. Shuaib, King Robert, Hudson Irene., (2014b). Transmuted Gompertz distribution with application" presented in the 8th Australia – New Zealand Mathematics Convention, held at Melbourne at Parkville Campus the University of Melbourne, from 8-12 December, 2014.
7. Merovci, F. (2013). Transmuted Rayleigh distribution. Austrian Journal of Statistics, Vol. 42: 1, 21-31.
8. Nicholas M. D. and Padgett W. J, (2006). A bootstrap control chart for Weibull percentiles, Quality and Reliability Engineering International. Vol. 22, 141-151.
9. W. Shaw and I. Buckley. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map. Research report, King's College, London, U.K.
10. Yuzhu Tian, Maozai Tian & Qianqian Zhu. (2014). Transmuted Linear Exponential Distribution: A New Generalization of the Linear Exponential Distribution, Communications in Statistics - Simulation and Computation, 43:10, 2661-2677.