Some convergence theorems for bivariate exponential dispersion models

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Abstract

The need of adequate regression models for multivariate non-normal correlated data has motivated the development of flexible multivariate distribution families. In a recent paper, Jørgensen and Martínez have presented an extension of Jørgensen's univariate exponential dispersion models using an adaptation of the convolution method in order to obtain models with a flexible correlation structure and marginal distributions of the same family. Induced by the convergence results found for the univariate case, it is speculated that these multivariate dispersion models can converge, under certain conditions, to a Tweedie probability distribution generated by the convolution method mentioned above, or to a gamma distribution as an extension of the theorem developed by Jørgensen, Martínez and Tsao for the univariate case in 1994. As a first step to a generalization to k dimensional case, it seems reasonable to be able to characterize the domain of attraction of the Tweedie models, particularly for the bivariate gamma models. Because of this, the aim of this work is to extend to $\mathbb{R}^2$ the results about the convergence of exponential dispersion models. Omey and Willekens present an extension of Tauber-Karamata Theorem for the bivariate case; this result will be useful to express the moment generating function of a multivariate exponential dispersion model in terms of regular variation function. This leads to try to extend the gamma convergence theorem developed by Jørgensen, Martínez and Tsao to bivariate case. In a first stage the work being carried out is the proof for the particular case where the regular variation order is the same for both variables, i.e. using the definition of regular variation of Stam.

Keywords: regular variation, Tauber theorem, gamma convergence.

1. Introduction

For decades the use of the normal probability distribution has been extensive in statistical data analysis, however it is known that for many data sets it is not appropriate. In 1972 Nelder and Wedderburn (1972) were the first to consolidate under a single approach a variety of non-normal variables involving both discrete and continuous variables but limiting them to the natural exponential family (NEF), introducing the generalized linear model class. Some years later Jørgensen(1997) extends this class of models to a more comprehensive class called dispersion models, covering a wider range of non-normal probability distributions. These models are characterized by the canonical parameter of the associated NEF and the dispersion parameter. The Tweedie models are a widely studied class of exponential dispersion (ED) models. They are very important, since the ED models that are closed with respect to scale transformations are Tweedie models and, on the other hand, it is noteworthy that many ED models can be approximated by Tweedie models. These important theoretical results are obtained by using the Tauberian theorems and regular variation condition of the variance functions.

The need of adequate regression models for multivariate non-normal correlated data has motivated the development of flexible multivariate distribution families. In a recent paper, Jørgensen and Martínez have presented an extension of ED univariate models. These authors constructed multivariate dispersion models by using an extended convolution method in order to obtain models with a flexible correlation structure and

marginal distributions of the same family. Induced by the convergence results found for the univariate case, it is speculated that these multivariate dispersion models can converge, under certain conditions, to a Tweedie probability distribution generated by the convolution method mentioned above, or to a gamma distribution as an extension of the theorem developed by Jørgensen, Martínez and Tsao in 1994 for the univariate case (Jørgensen, 1997).
As a first step to a generalization to k-dimensional case, the aim of this work is to extend to $\mathbb{R}^2$ the results about the convergence of ED univariate models.

2. Multivariate models

Consider a multivariate NEF for the k-dimensional vector $\mathbf{Z}$ with parameter $\boldsymbol{\theta}$ with domain
$\boldsymbol{\Theta} = \left\{\boldsymbol{\theta} \in \mathbb{R}^{\mathbf{k}} : \int a\left(\mathbf{z}\right) e^{\mathbf{z}^T \boldsymbol{\theta}} d\mathbf{z} < \infty\right\}$, cumulant function κ, mean vector μ and covariance matrix $V(\boldsymbol{\mu})$ for $\boldsymbol{\mu} \in \boldsymbol{\Omega} = \dot{\kappa}(\text{int}\boldsymbol{\Theta})$.
The *additive exponential model* generated by the multivariate NEF is defined by the probability density function

$$f^*(\mathbf{z}; \boldsymbol{\theta}, \lambda) = a^*(\mathbf{z}; \lambda) exp\left[\mathbf{z}^T\boldsymbol{\theta} - \lambda\kappa(\boldsymbol{\theta})\right] \text{ para } \mathbf{z} \in \mathbb{R}^k \tag{1}$$

for some function $a^*(\mathbf{z}; \lambda)$, obtained by replacing the cumulant function κ with $\lambda\kappa$. It is assumed that (1) is infinitely divisible, such that λ has domain $\mathbb{R}_+$. The mean vector (1) is $\lambda\boldsymbol{\mu}$ and the covariance matrix $\lambda V(\boldsymbol{\mu})$ where V is called the unit variance function. The model with k+1 parameters corresponding to (1) is denoted by $ED^*(\boldsymbol{\mu}, \lambda)$. The *reproductive exponential dispersion model* is defined applying the duality transformation $\mathbf{Y} = \mathbf{Z}/\lambda$ to (1), giving

$$f(\mathbf{y}; \boldsymbol{\theta}, \lambda) = a(\mathbf{y}; \lambda) exp\left\{\lambda\left[\mathbf{y}^T\boldsymbol{\theta} - \kappa(\boldsymbol{\theta})\right]\right\} \text{ para } \mathbf{y} \in \mathbb{R}^k, \tag{2}$$

for some $a(\mathbf{y}; \lambda)$. A vector $\mathbf{Y}$ distributed according to (2) has mean $\boldsymbol{\mu}$ and covariance matrix $Var(\mathbf{Y}) = \lambda^{-1}V(\boldsymbol{\mu})$. It can be seen that the last expression is governed by a single parameter λ. Because of that Jørgensen and Martínez (2012) modify it in order to obtain a more flexible covariance structure. This generalization is presented below for the bivariate case.

Bivariate case

The starting point is a bivariate NEF with cumulant function $\kappa\left(\boldsymbol{\theta}\right) = \kappa(\theta_1, \theta_2)$. Let be s and t the arguments of the cumulate generating function (CGF)

$$(s, t) \longmapsto \lambda_{12}\kappa_\theta(s, t). \tag{3}$$

where the parameter λ_{12} has as domain $\mathbb{R}_+$ due to the infinitely divisible assumption. The method is based on the following vector representation $\mathbf{Z} = (Z_1, Z_2)^T$,

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ U_2 \end{bmatrix} + \begin{bmatrix} U_{11} \\ U_{12} \end{bmatrix}, \tag{4}$$

where the right side vectors of (4) are assumed independent, $U_j \sim ED^*(\mu, \lambda_j),\ j = 1, 2$, the joint distribution of U_{11} and U_{12} has CGF given by (3) and the marginal distributions of U_{11} and U_1 belong to the same family $ED^*(\mu_1, \lambda_1)$, the same way that marginal distributions of U_{12} and U_2 belong to the same family, $ED^*(\mu_2, \lambda_2)$. The CGF for the vector $\mathbf{Z}$ is given by:

$$K_{\boldsymbol{\theta}}(s, t) = \lambda_{12}\kappa_{\boldsymbol{\theta}}(s, t) + \lambda_1\kappa_{\boldsymbol{\theta}}(s, 0) + \lambda_2\kappa_{\boldsymbol{\theta}}(0, t). \tag{5}$$

Let $\dot{\kappa}_1(\theta_1, \theta_2)$ and $\dot{\kappa}_2(\theta_1, \theta_2)$ be the first order derivatives and $\ddot{\kappa}_{ij}(\theta_1, \theta_2)$ for $i, j = 1, 2$ the second order derivatives of κ. Then:

$$E(\mathbf{Z}) = \begin{bmatrix} (\lambda_{12} + \lambda_1)\,\dot{\kappa}_1(\theta_1, \theta_2) \\ (\lambda_{12} + \lambda_2)\,\dot{\kappa}_2(\theta_1, \theta_2) \end{bmatrix} = \begin{bmatrix} \lambda_{11}\mu_1 \\ \lambda_{22}\mu_2 \end{bmatrix},$$

where μ_j are the components of $\boldsymbol{\mu}$ and

$$Cov(\mathbf{Z}) = \left[\begin{array}{cc} \lambda_{11}\ddot{\kappa}_{11}(\theta_1,\theta_2) & \lambda_{12}\ddot{\kappa}_{12}(\theta_1,\theta_2) \\ \lambda_{12}\ddot{\kappa}_{21}(\theta_1,\theta_2) & \lambda_{22}\ddot{\kappa}_{22}(\theta_1,\theta_2) \end{array}\right] = \left[\begin{array}{cc} \lambda_{11}V_{11}(\boldsymbol{\mu}) & \lambda_{12}V_{12}(\boldsymbol{\mu}) \\ \lambda_{12}V_{21}(\boldsymbol{\mu}) & \lambda_{22}V_{22}(\boldsymbol{\mu}) \end{array}\right],$$

where $V_{ij}(\boldsymbol{\mu})$, $i=1,2\, j=1,2$ are the components of the unit variance function 2x2 $V(\boldsymbol{\mu})$. The reproductive form of the model for the vector $\mathbf{Y}$ defined as

$$\mathbf{Y} = \left[\begin{array}{c} Y_1 \\ Y_2 \end{array}\right] \quad = \quad \left[\begin{array}{c} Z_1/\lambda_{11} \\ Z_2/\lambda_{22} \end{array}\right],$$

has mean vector $E(\mathbf{Y}) = (\mu_1,\mu_2)^T$ and covariance matrix

$$\mathrm{Cov}(\boldsymbol{Y}) = \left[\begin{array}{cc} \frac{1}{\lambda_{11}}V_{11}(\boldsymbol{\mu}) & \frac{\lambda_{12}}{\lambda_{11}\lambda_{22}}V_{12}(\boldsymbol{\mu}) \\ \frac{\lambda_{12}}{\lambda_{11}\lambda_{22}}V_{21}(\boldsymbol{\mu}) & \frac{1}{\lambda_{22}}V_{22}(\boldsymbol{\mu}) \end{array}\right] = \boldsymbol{\Sigma} \odot V(\boldsymbol{\mu}).$$

This construction provides the bivariate ED model with the desired flexible correlation structure.
Particular forms of the moment generating functions (MGF) of the univariate ED models that involve a slowly variation function allow to proof the convergence properties to limiting distributions. In this regard Omey and Willekens (1989) introduce two extensions of the regular variation definition for $\mathbb{R}^k$. The particular case for $\mathbb{R}^2$ is presented in the next section.

3. Bivariate regular variation

Definition 1. A measurable function $u : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ is said to *vary regularly at infinity* with exponents α and $\beta \in \mathbb{R}$ if, for all $x\,, y > 0$,

$$\lim_{\min(t,s)\to\infty} \frac{u(tx,sy)}{u(t,s)} = x^\alpha y^\beta$$

exists and it is finite, being $u \in RV\,(\alpha,\beta)_{\,\infty}$.

Remark. If $\alpha=\beta$ and $t=s$, the regular variation definition simplifies the expression below:

$$\lim_{t\to\infty} \frac{u(tx,ty)}{u(t,t)} = (xy)^\alpha\,,$$

that leads to the Stam's regular variation definition (Stam, 1977) with a single variation order for both variables.

In accordance to slow variation function definition for the univariate case, it is extended to $\mathbb{R}_+^2$ in the following way:

Definition 2. A function $L(x,y)$ is said to *vary slowly at infinity* if, for any $x,\ y>0$:

$$\lim_{\min(t,s)\to\infty} \frac{L(tx,sy)}{L(t,s)} = 1.$$

And it is said to *vary slowly at zero* if, for any $x,\ y>0$

$$\lim_{\min(t,s)\to\infty} \frac{L\left(\frac{t}{x},\frac{s}{y}\right)}{L\,(t,s)} = 1.$$

Notation: $L \in LV_{\,\infty(0)}$.

Proposition 3. *$u\,(x,y) \in RV\,(\alpha,\beta)_{\,\infty(0)} \Leftrightarrow u(x,y) = x^\alpha y^\beta L(x,y)$ where $L \in LV_{\,\infty(0)}$.*

Omey and Willekens (1989) also present an extension of the Tauber-Karamata theorem for the bivariate case, result that will be useful to express the MGF of a multivariate ED model in terms of regular variation functions.

4. Tauber-Karamata Theorem

Omey and Willekens enunciate the Tauber-Karamata theorem as:

Theorem 4. *Let be ν a measure on $\mathbb{R}_+^2$ with Laplace transform finite in $\mathbb{R}_+^2$, the function $\bar{\nu}$ defined in section 3 belongs to $RV\left(\alpha,\beta\right)_{\infty(0)}$ if and only if the corresponding Laplace transform $\omega\left(1/t, 1/s\right) \in RV\left(\alpha,\beta\right)_{\infty(0)}$ being α and β non-negative. Furthermore, in such case*

$$\lim_{\min(t,s)\to\infty} \frac{\omega\left(x/t, y/s\right)}{\omega\left(t,s\right)} = \frac{\Gamma\left(1+\alpha\right)\Gamma\left(1+\beta\right)}{x^{\alpha}y^{\beta}}.$$

An equivalent statement, can be obtained through a generalization over the Jørgensen's statement in (Jørgensen, 1997) for the univariate case.

Theorem 5. *Let ν be a measure over $\mathbb{R}_+^2$ with Laplace transform $\omega\left(t,s\right)$, $L \in LV_{\infty(0)}$ and let α and β non-negative. Then*

$$\bar{\nu}\left(t,s\right) \sim \frac{1}{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)} t^{\alpha}s^{\beta}L\left(t,s\right) \Longleftrightarrow \omega\left(\frac{1}{t},\frac{1}{s}\right) \sim t^{\alpha}s^{\beta}L\left(t,s\right)$$

when $\min\left(t,s\right)\to\infty\left(0\right)$, *where $\bar{\nu}$ is the function defined in section 3.*

It could be proved that the statements from theorem 5 are equivalent to those in theorem 4.
Consider ν a measure of the form $\nu\left(\mathsf{d}x,\mathsf{d}y\right) = g\left(x,y\right)x^{\alpha-1}y^{\beta-1}$, with g analytic and non-null in $(0,0)$, so that $\upsilon \in RV_{(0)}$ with exponents α and β. Taking into account the statements from Tauber-Karamata theorem, it can be said that the CGF of the NEF generated by the same measure has the form

$$M_{\nu}(\theta_1,\theta_2) = \left(-\theta_1\right)^{-\alpha}\left(-\theta_2\right)^{-\beta}L\left(-\theta_1,-\theta_2\right),\ \theta_1,\theta_2 < 0,$$

where $L\left(-\theta_1,-\theta_2\right) \in LV_{\infty}$.

5. Convergence results in $\mathbb{R}^2$

The possibility to express the CGF as a product of powers of the canonical parameters and a slow varying function allows to extend the gamma convergence theorem developed by Jørgensen, Martínez and Tsao in 2004 to the bivariate case. This theorem states that given a $ED(\mu,\sigma^2)$ model with mean domain Ω and index set Λ generated by a measure ν that varies regularly at zero or at infinite with exponent γ, it is fulfilled that for any $\mu > 0$ and any σ^2 such that $1/\sigma^2 \in \Lambda$:

$$\frac{1}{c}ED(c\mu,\sigma^2) \xrightarrow{d} Ga\left(\mu,\sigma^2/\gamma\right)$$

for c tending to zero or to infinite, respectively. In a first stage the work being carried out is the proof for the particular case in which the regular variation order is the same for both variables ($\alpha=\beta$), i.e. using Stam's definition of regular variation (Stam, 1977). For this case the $ED(\boldsymbol{\mu},\boldsymbol{\Sigma})$ model generated by the measure ν of regular variation at zero or at infinite with exponent $\alpha > 0$ for both variables verifies by Tauber-Karamata theorem extension that the MGF for ν takes the form

$$M_{\nu}(\theta_1,\theta_2) = \left(\theta_1\theta_2\right)^{-\alpha}L\left(-\theta_1,-\theta_2\right),\ \theta_1,\theta_2 < 0.$$

and it is conjectured that $\frac{1}{c}ED(c\boldsymbol{\mu},\boldsymbol{\Sigma})$ converges in distribution, when c tends to zero or infinite respectively, to a bivariate gamma distribution due to Kibble and Moran (Kotz *et al*, 2000).

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