A new framework for extending the Wishart distribution

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Abstract

The Wishart distribution and its generalizations are among the most prominent probability distributions in multivariate statistical analysis, arising naturally in applied research and as a basis for theoretical models. In this paper, we generalize the Wishart distribution utilizing a different approach that leads to the Wishart generator distribution with the Wishart distribution as a special case. Important statistical characteristics of the Wishart generator distribution are derived from the matrix theory viewpoint. Estimation is also touched upon as a guide for further research from the classical approach as well as from the Bayesian paradigm. The paper is concluded by giving applications of two special cases of this distribution in calculating the product of beta functions and astronomy.

Keywords: estimation; generator; eigenvalue; zonal polynomial.

1. Introduction

The Wishart distribution and its generalizations are among the most prominent probability distributions in multivariate statistical analysis, arising naturally in applied research and as a basis for theoretical models. The reader is referred to Gupta and Nagar (2000) and Anderson (2003) for a more extensive study regarding the theoretical as well as the practical uses of the Wishart distribution. Various generalizations and extensions are proposed for the Wishart distribution, because of its importance in matrix theory. To mention a few: Sutradhar and Ali (1989) generalized the Wishart distribution for the vector variate elliptical models, however Teng et al. (1989) considered matrix variate elliptical models in their study. Wong and Wang (1995) defined the Laplace-Wishart distribution. In the context of graphical models, Roverato (2002) defined the hyper-inverse Wishart and Wang and West (2009) extended the inverse Wishart distribution for using hyper-Markov properties (see Dawid and Lauritzen, 1993), while Bryc (2008) proposed the compound Wishart and q-Wishart in graphical models. Abul-Magd (2009) proposed a generalization to Wishart-Laguerre ensembles. Adhikari (2010) generalized the Wishart distribution for probabilistic structural dynamics, and Díaz-García and Gutiérrez-Jáimez (2011) extended the Wishart distribution for real normed division algebras. Munilla and Cantet (2012) also formulated a special structure for the Wishart distribution to apply in modeling the maternal animal. For a recent extension, see Carlo-Lopera et al. (2014).There are of course many extensions that are not listed in the above; however the Wishart distribution can be viewed in the sense that it gives rise to other distributions. In this paper, a metamorphosis in the structure of the density of Wishart distribution is proposed, which opens a new era of constructing matrix variate distributions for symmetric positive definite matrices. Some statistical characteristics of this distribution are derived. Estimation is also touched upon as a guide for further research from the classical approach as well as from the Bayesian paradigm. The paper is concluded by proposing further developments beyond the Wishart generator distribution.

2. Wishart generator distribution

In this section a new family of distributions namely the Wishart Generator Distribution (WGD) is defined and some special cases along with the definition of inverse WGD are given. In a nutshell, the

new generator type distribution concludes from a special case of Teng et al (1989). This distribution is named as such since the Wishart distribution is a member of this family. Some other members are the matrix variate t distribution, the power Wishart distribution and the logarithmic Wishart distribution amongst others.

3. Properties

In this section we derive some important statistical properties of the WGD :

- The moments;
- Characteristic function;
- Laplace transform;
- Joint density function of eigenvalues;
- Density function of largest eigenvalue;
- Density function of trace;
- Ratios – here the reader is also referred to the work of Bekker et al (2011a ; 2011b)

4. Estimation

In this section, we briefly consider some estimation methodologies for the parameter of the WGD, including the classical (maximum likelihood estimator) and Bayesian viewpoints.

5. Conclusions

Our main contribution is the introducing of the Wishart generator distribution which includes the Wishart as a special case; thus a variety of matrix variate distributions is described under this unified framework. We provided the reader with some plausible extensions of WG distribution and briefly considered some applications. The Wishart generator distribution might be important for a number of practical signal processing applications including synthetic aperture radar (SAR), multi-antenna wireless communications and direct imaging of extra-solar planets.

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