Optimal Periodic Maintenance Policy under the Arithmetic Reduction Age Imperfect Repair Model

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Abstract
This paper presents the preliminary results of a research work on procedures for the determination of optimal maintenance policies under imperfect repair assumption. The topic was motivated by a data set on failures of off-road trucks used by a mining company. In particular, it is discussed the determination and practical implementation of an optimal preventive maintenance policy using a virtual age model. Under such imperfect repair models, the expected number of failures (or mean function) at time t is given by a general renewal function with no closed form solution available. In this work, a procedure to approximate the mean function is proposed. Optimal periodic maintenance policies are obtained for the off-road trucks engines, providing information for the maintenance decision-making in these equipment.

Keywords: reliability; imperfect repair; virtual age model; optimal stopping.

1. Introduction
This paper was motivated by a real situation concerning failures in off-road trucks used by a Brazilian mining company. The reliability of these trucks depends on the reliability of many of its components such as the engine, the weighbridge, the tracking system and the cockpit. The problem under study refers only to diesel engines. The data used refers to a sample of 193 such engines, and among them, 52 were right censored, since their last inspection time corresponded to a removal for a Preventive Maintenance (PM) action. The accumulated number of working hours was registered, as well as the number of hours at which each preventive maintenance or failure took place. 208 failure times were recorded, with some engines presenting more than one failure (at most 4 per system) and some no failure at all.

Figure 1 plots the mean cumulative number of failures versus time (in hours of operation) for the sample of 193 engines. It is of interest to avoid a break in these engines, and consequently to reduce the necessity for corrective maintenance, which has higher cost (on average approximately 23% larger) than the PM. Hence, the company needs to adopt a maintenance policy that favors PM actions, as opposed to repair actions taken after failures.

The approach proposed in this paper takes into account the effect of repair actions implemented after each failure (repair efficiency) in order to determine the optimal periodicity of PMs. The PM actions are supposed here to be perfect.

In the maintenance literature one can find a great number of papers that propose PM policies under the assumption of Minimal Repair (MR) on failures. However, in many practical situations, more realistic notions of repair, intermediate between the extremes perfect and minimal repair, might be needed. In other words, many repairs actions are more likely to be Imperfect Repairs (IR), and any attempt to elaborate an optimal maintenance policy must take the actual degree of efficiency of these repairs into account.

Many models have already been proposed for IR effects. Among them are the virtual age models proposed by Kijima et al. (1988), where the degree of efficiency of the repair is represented by the parameter $\theta$ ($0 \leq \theta \leq 1$) and includes minimal and perfect repair as special cases ($\theta = 1$ and $\theta = 0$, respectively). However, under IR
assumption there is no closed form expression to find the optimal PM periodicity. To overcome this difficulty, an approximation procedure was proposed by the authors, but its usage depends on the knowledge of the repair efficiency ($\theta$) and the distribution of the lifetimes of the systems being studied.

Only a few IR models have been statistically studied, and the emphasis has been on the estimation of the repair efficiency parameter. Doyen and Gaudoin (2004) proposed two new classes of IR models. In the first class of models, the repair effect is expressed by a reduction in the failure intensity (the so called Arithmetic Reduction of Intensity or ARI models). In the second class, the repair effect is expressed by a reduction in the system’s virtual age (the so called Arithmetic Reduction of Age or ARA models). The virtual age model proposed by Kijima et al. (1988) corresponds to the particular case of ARA model with memory 1, namely, ARA$_1$.

This paper presents preliminary results from a study with two purposes: (1) use the history of failure times of the engines to estimate statistically the effect of IRs, under an ARA$_1$ model, and (2) giving that information, find the optimal PM policy.

From a modelling point of view, $\{N(t)\}_{t \geq 0}$ (where $N(t)$ denotes the number of observed failures up to time $t$) is a stochastic point process, with mean function $\Phi(t) = E[N(t)]$ and failure intensity function

$$\lambda(t) = \lim_{\delta t \to 0} \frac{P(N(t + \delta t) - N(t) = 1 | \mathcal{F}_t)}{\delta t}, \forall t \geq 0$$

(1)

where $\mathcal{F}_t$ represents the history up to time $t$ (informally, one could think of $\mathcal{F}_t$ as the information provided by the failure times $0 < t_1 < \cdots < t_{N(t)} < t$).

An important function associated to the intensity function is the rate of occurrence of failures (ROCOF), given by $\phi(t) = E[\lambda(t)]$, $t \geq 0$. It can be shown (see, e.g. Aalen, 1978 and Aalen et al., 2008) that

$$\Phi(t) = \int_0^t E[\lambda(s)] \, ds.$$

Under the MR assumption, it is assumed that the effect of repair is to leave the system in the same state as it was just before failure. The underlying failure process in this case is a Nonhomogeneous Poisson Process (NHPP), and the failure intensity function $\lambda(t)$ equals the ROCOF.

Under the IR assumption, some functional forms for $\lambda(t)$ have been proposed in the literature. In particular, for ARA$_1$ model, it can be shown that the failure intensity is given by (Doyen and Gaudoin, 2004)

$$\lambda(t) = \lambda_R(t - (1 - \theta)T_{N(t)}),$$

(2)

where $T_n$ is a random variable representing the real age of the system at the $n$th failure (the elapsed time since the initial start-up of the system) and $\lambda_R(t)$ is the reference initial intensity function.

If the intensity function is the one given in Equation (2), it can be shown (Kijima et al., 1988) that the expected number of failures (or mean function) at time $\tau$ is given by

$$\Phi(\tau) = \int_0^\tau E[\lambda_R(t - (1 - \theta)T_{N(t)})] \, dt.$$
This equation is usually referred to in the literature as the *general renewal function*. Unfortunately, there is no closed form solution for this equation, except for special cases, such as \( \theta = 1 \) (MR), or \( \theta = 0 \) with the underlying failure times exponentially distributed.

In view of the limitations of the approximations cited above, this paper proposes a procedure to obtain estimators for Equation (3) using the observed failure history. The proposed method aims at dealing with the following three issues at the same time, namely: (1) the estimation of the parameters of the intensity function \( \lambda(t) \) (Equation (2)), (2) the calculation of an estimator for the mean function \( \Phi(t) \) (Equation (3)), and (3) the combination of (1) and (2) to find the optimal PM policy, *i.e.*, to obtain the optimal PM check points (or periodicity \( \tau \)) that minimize the expected total cost (preventive and corrective maintenance actions) under an IR environment. The method is applied to the failure history of the off-road engines.

The outline of this paper is as follows. In Section 2, the cost function to be minimized is presented. A method to find the optimal PM policy (given by check points at every \( \tau \) units of time) is proposed. The method is applied to the off-road engines maintenance data and the results are presented in Section 3 (point and interval estimates for \( \tau \) are provided). Conclusions and final comments end the paper in Section 4.

2. Cost Function and Optimal PM Under an ARA\(_1\) Model

Consider a system which is subject to failure, and that is put in operation at time \( t = 0 \). Assume the following conditions:

- PM check points are scheduled after every \( \tau \) units of time;
- at each PM check point, a maintenance of fixed cost \( C_{PM} \) takes place, which instantly returns the system to the AGAN condition;
- between successive PM check points, an IR of degree \( \theta \) \( (0 \leq \theta \leq 1) \) is executed after each failure, where \( \theta = 1 \) represents an MR (ABAO condition) and \( \theta = 0 \) a perfect repair (AGAN);
- the expected cost for each IR action is \( C_{IR} \);
- repair costs and failure times are independent;
- repair times are neglected.

Assume that PM is performed every \( \tau \) units of time. The expected maintenance cost per unit time for the system is given by (Gilardoni and Colosimo, 2007)

\[
C(\tau) = \frac{C_{PM} + C_{IR}E[N(\tau)]}{\tau}, \quad \tau > 0.
\] (4)

Under ARA\(_1\) model, \( E[N(\tau)] \) is given by Equation (3). The objective here is to find an optimal PM interval that minimizes Equation \( C(\tau) \), which is the value \( \tau \) that satisfies

\[
D(\tau) = \tau\phi(\tau) - \Phi(\tau) = \frac{C_{PM}}{C_{IR}},
\] (5)

where \( \phi(\tau) = \frac{d}{d\tau}\Phi(\tau) \) is the ROCOF function for the system.

However, under IR assumption no closed form solution can be obtained for \( \Phi(\tau) \) and, consequently, for Equation (5). So, we combine the estimation of parameters from Equation (2) and from the mean function in (3) to obtain the optimal PM periodicity, as the solution for Equation (5).

The mean function \( \Phi(t) \) is estimated using a combination of Maximum Likelihood Estimates (MLE) for the parameters involved \((\theta, \beta, \eta)\), Monte Carlo simulation, and the Nelson–Aalen nonparametric procedure (Aalen, 1978), also known as the Mean Cumulative Function (MCF). The proposed method is illustrated using the PLP but it can be applied to any other parametric form chosen for the initial intensity.

Appropriate confidence intervals for \( \tau \) are obtained using nonparametric bootstrap resampling (Efron and Tibshirani, 1986). The method consists of resampling with replacement \( B \) (\( B \) large) samples from the original
database, each resample having the same size as the original data set.

4. Off-Road Engines Maintenance Data Revisited

In this section, we return to the situation described in Section 1, i.e., the off-road engines maintenance data presented in Figure 1. In that figure, the mean observed cumulative number of failures showed a convex shape, indicating that the intensity function is increasing for these systems, therefore justifying PM. The analyses were done using the R language. The goal here is to obtain the optimal PM check points that minimize the expected total cost. The results are summarized in Figure 2 and Table 1. The main points are listed below:

- MR assumption: The following point and interval (95% confidence intervals based on Normal approximation) MLEs were obtained for the PLP parameters: \( \hat{\beta} = 2.126 \) (1.916; 2.358) and \( \hat{\eta} = 16.715 \) (15.604; 17.905). Figures 2 (a) and (b) (dotted lines) show the MLEs of the mean function and ROCOF respectively. These estimates were obtained using the invariance property of the MLEs. Next, following Gilardoni and Colosimo (2007) approach, \( \hat{\Phi} \) and \( \hat{\phi} \) were then plugged into Equation (5). Figure 2 (c) (dotted line) exhibits the function \( \hat{D}(t) \).

- IR assumption: MLEs and 95% CIs were obtained for the PLP parameters (\( \hat{\beta} = 2.458 \) (2.185; 2.765) and \( \hat{\eta} = 15.586 \) (14.605; 16.633)) and the effect of repair parameter (\( \hat{\theta} = 0.471 \) (0.330; 0.673)) (Step 1). Next we follow Steps 2 and 3 to estimate the mean function \( \Phi(t) \), defined in Equation (3), which does not have a closed solution. Firstly, the MCF was obtained from \( K = 10,000 \) failure processes simulated with the truncation time \( T = 40,000 \) h (that corresponds to the time range in the observed data). Secondly, the Greatest Convex Minorant (GCM) of the MCF provided an estimation for \( \Phi(t) \) (Figure 2 (a), solid line), whose derivation generated an estimation for \( \phi(t) \), the ROCOF of this process (Figure 2 (b), solid line). After replacing these estimated functions in Equation (5), the function \( \hat{D}(t) \) was obtained (Figure 2 (c), solid line).

It is noteworthy that the estimated value for \( \theta \) and the corresponding confidence interval suggest that the repair actions after failures are neither minimal (\( \theta = 1 \)) nor perfect repairs (\( \theta = 0 \)). Therefore, the traditional modeling MR assumption is inappropriate for the off-road engines, reassuring one as to the relevance of the methodology proposed in this paper. Therefore, any conclusions taken under the MR assumption are subject to bias.

The function \( D(t) \) provides the optimal PM check points. According to Equation (5), the policy that minimizes the expected total cost is the value \( \tau \) that satisfies \( D(\tau) = \frac{C_{PM}}{C_{IR}} \). Figure 2 (c) shows the graphs of this function under MR (dotted line) and IR (solid line) assumptions. Interpolations may be done fixing values for \( \frac{C_{PM}}{C_{IR}} \) in y-scale, what provides estimations for \( \tau \) in x-scale. Recall that according to the mining company, the cost of a corrective maintenance performed after an unexpected failure is 23% higher than the cost of preventive maintenance. Therefore, using the ratio \( \frac{C_{PM}}{C_{IR}} = 1/1.23 \), the optimal maintenance periodicity obtained under IR was 15,815 hours (or approximately 659 days), while the optimal periodicity obtained under MR was 14,345 hours (around 598 days).

<table>
<thead>
<tr>
<th>( \frac{C_{PM}}{C_{IR}} )</th>
<th>( \hat{\tau}_{MR} ) CI</th>
<th>( \hat{\tau}_{IR} ) CI</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1.23</td>
<td>(13,304; 15,511)</td>
<td>(18,082)</td>
</tr>
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So, it can be concluded that, for the off-road engines data, the optimal PM periodicity is 15,815 hours, with 95% confidence interval from 13,632 to 18,082 hours. For a new system, this is the time that must be expected for the first PM action.

5. Final Remarks
In this paper, an optimal preventive maintenance (PM) check point was obtained for the case of repairable systems subject to perfect preventive maintenance actions (which returns them to an AGAN condition) and imperfect repairs (IR) after a failure. The IRs were assumed to be of degree $\theta$ ($0 \leq \theta \leq 1$, unknown), following an ARA$_1$ model. The motivating practical situation concerned failure histories of off-road truck engines used by a mining company.

The estimates of the model parameters were obtained jointly by the Maximum Likelihood method, namely, the PLP parameters and the degree of repair $\theta$.

Next, a method for estimating the mean function $\Phi(t)$ under an ARA$_1$ model was presented. The method combined Monte Carlo simulation and the calculation of the (nonparametric) Mean Cumulative Function. This procedure made it possible to estimate the optimal preventive maintenance check point ($\tau$) for the practical situation under study. Confidence intervals were also obtained for this quantity, using bootstrap resampling. The results were compared with the ones obtained under a minimal repair assumption.

Recall that, by using an ARA$_1$ model, the main challenge here was not only to come up with a good approximation to the g-renewal function involved in the expression of the Mean Function ($\Phi(t)$), but one that was differentiable. With this in mind, one solution was to use the Greatest Convex Minorant of the MCF.

Figure 2: Estimated functions for the off-road engines data, under MR (dotted lines) and IR (solid lines) assumptions: (a) $\hat{\Phi}(t)$, (b) $\hat{\phi}(t)$ and (c) $\hat{D}(t)$ (Equation (5)), versus time.
As far as the practical results are concerned, two important pieces of information were provided to the mining company: (1) the degree of repair: the estimated value of $\theta$ and the corresponding confidence interval indicated that the repair actions after failures are neither minimal nor perfect repairs; (2) optimal preventive maintenance check points: in order to minimize the total expected maintenance cost, the mining company needs to implement PM after every 15,815 h, i.e., around 22 months.

**References**


