Preservation of non-parametric distributions classes under a system signature point process

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Abstract

We analyse the preservation properties of increasing failure rate distributions relative to $(\Im_t)_{t\geq 0}$ and new better than used distributions relative to $(\Im_t)_{t\geq 0}$ distributions classes, as in Arjas (1981), under a point process signature representation of a coherent system.

Keywords: increasing failure rate distribution relative to $(\Im_t)_{t\geq 0}$; new is better than used distribution relative to $(\Im_t)_{t\geq 0}$.

1. Introduction

Classes of non-parametric distributions, such as increasing failure rate (IFR) distributions, new better than used (NBU) distributions and others, have been extensively investigated in Reliability Theory. They can be used to achieve the benefit of a maintenance operation or to derive bounds on system reliability.
Several extensions of these concepts appeared in the literature. Arjas (1981), considered to observe the components continuously in time, based on a family of sub σ-algebras $(\Im_t)_{t\geq 0}$, to introduce the notion of multivariate increasing failure rate distribution and multivariate new better than used distribution relative to σ-algebras $(\Im_t)_{t\geq 0}$, denoted by MIFR$|\Im_t$ and MNBU$|\Im_t$, respectively, generalizing the conventional definition of IFR and NBU. We intend to analyse the preservation properties of these non-parametric distribution classes for a coherent system under a point process signature representation. The coherent system reliability representation through the system signature, as in Samaniego (1985), is substantially useful:

Definition 1.1 Let T be the lifetime of a coherent with component lifetimes $T_1, ..., T_n$ which are independent and identically distributed (i.i.d.) random variables with continuous distribution F. Then the signature vector is defined as $\mathbf{s} = (s_1, ..., s_n)$ where $s_i = P(T = T_{(i)})$ and $\{T_{(i)}, 1 \leq i \leq n\}$ are the order statistics of $\{T_i, 1 \leq i \leq n\}$.
Under the definition assumptions we have

$$P(T > t) = \sum_{i=1}^{n} P(T = T_{(i)}) P(T_{(i)} > t | T = T_{(i)}) = \sum_{i=1}^{n} P(T = T_{(i)}) P(T_{(i)} > t).$$

A detailed treatment of the theory and applications of system signature may be found in Samaniego (2007). Bueno (2013) define the signature point process to introduce the above structure under a more general condition. Under a complete information level $(\Im_t)_{t\geq 0}$ we have

$$P(T > t|\Im_t) = \sum_{i=1}^{n} 1_{\{T=T_{(i)})\}} 1_{\{T_{(i)}>t\}}.$$

This representation is suitable to analyse coherent systems of dependent components. In this context, to generalize classical results using the properties of non-parametric classes of distributions, we need to verify if these properties are preserved under such representation.
In Section 2 we discuss the point process approach and the Arjas (1981) concepts of conditioned distribution classes. In Section 3 we analyse preservation results of the classes MIFR$|\Im_t$ and MNBU$|\Im_t$ under a system signature point process. As an application, in Section 4, we study the point process signature representation of the system under an event in time.

2. Section 2
2 .Preliminaries.

2.1. Signature point processes
In our general setup, we consider the vector $(T_1, ..., T_n)$ of n component lifetimes which are finite and positive random variables defined in a complete probability space $(\Omega, \Im, P)$, with $P(T_i \neq T_j) = 1$, for all $i \neq j, i, j$ in $C = \{1, ..., n\}$, the index set of components. The lifetimes can be dependent but simultaneous failures are ruled out.
The mathematical description of our observations, the complete information level, is given by a family of sub σ-algebras of $\Im$, denoted by $(\Im_t)_{t\geq 0}$, where

$$\Im_t = \sigma\{1_{\{T_{(i)}>s\}}, X_i = j, 1 \leq i \leq n, j \in C, 0 < s \leq t\},$$

satisfies the Dellacherie conditions of right continuity and completeness .
Intuitively, at each time t the observer knows if the event $\{T_{(i)} \leq t, X_i = j\}$ have either occurred or not and if it had, he knows exactly the value $T_{(i)}$ and the mark X_i. Follows that the component and the system lifetimes T are $\Im_t$ stopping times.
The evolution of components in time define a marked point process given through the failure times and the corresponding marks. We denote by $T_{(1)} < T_{(2)} < ... < T_{(n)}$ the ordered lifetimes $T_1, T_2, ..., T_n$, as they appear in time and by $X_i = \{j : T_{(i)} = T_j\}$ the corresponding marks. As a convention we set $T_{(n+1)} = T_{(n+2)} = ... = \infty$ and $X_{n+1} = X_{n+2} = ... = e$ where e is a fictitious mark not in C,considering the sequence $(T_n, X_n)_{n\geq 1}$ as a marked point process.

In what follows we assume that relations between random variables and measurable sets, respectively, always hold with probability one, which means that the term P-a.s., is suppressed. We consider the lifetimes $T_{(i),j}$ defined by the failure event $\{T_{(i)}, X_i = j\}$ with their sub-distribution function $F_{(i),j}(t) = P(T_{(i),j} \leq t) = P(T_{(i)} \leq t, X_i = j)$ suitable standardized.
Under the complete information level the behavior of the process $P(T > t|\Im_t)$, as the information flows continuously in time is given by the following Theorem

Theorem 2.1.1 Let $T_1, T_2, ..., T_n$ be the component lifetimes of a coherent system with lifetime T. Then,

$$P(T \leq t|\Im_t) = \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j}\leq t\}}.$$

Proof From the total probability rule we have

$$P(T \leq t|\Im_t) = \sum_{k,j=1}^{n} P(\{T \leq t\} \cap \{T = T_{(k),j}|\Im_t) = \sum_{k,j=1}^{n} E[1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j}\leq t\}}|\Im_t].$$

As T and $T_{(k),j}$ are $\Im_t$-stopping time, the event $\{T = T_{(k),j}\} \in \Im_{T_{(k),j}}$ where

$$\Im_{T_{(k),j}} = \{A \in \Im_\infty : A \cap \{T_{(k),j} \leq t\} \in \Im_t, \forall t \geq 0\},$$

we conclude that $\{T = T_{(k),j}\} \cap \{T_{(k),j} \leq t\}$ is $\Im_t$-measurable. Therefore

$$P(T \leq t|\Im_t) = \sum_{k,j=1}^{n} E[1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j}\leq t\}}|\Im_t] = \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j}\leq t\}}.$$

The above decomposition allows us to define the signature process at component level.

Definition 2.1.2 The marked point signature process of a coherent system with lifetime T is

$$(1_{\{T=T_{(k),j}\}}, 1 \leq k, j \leq n).$$

Remark 2.1.3 Clearly we have

$$P(T > t|\Im_t) = \sum_{k,j=1}^{n} 1_{\{T=T_{(k),j}\}} 1_{\{T_{(k),j}>t\}}.$$

2.2. Conditional Classes of distribution.

We let θ_t be a shift in time defined by

$$\theta_t T_i = (T_i - t)^+ = \max\{T_i - t, 0\}, \quad 1 \leq i \leq n.$$

We may think of $\theta_t T_i$ as the residual lifetime of T_i at time t. Let $\theta_t \mathbf{T} = (\theta_t T_1, .., \theta_t T_n)$.

Arjas (1981) introduced the following concepts of classes of non-parametric distributions based on conditional stochastic order.

Definition 2.2.1 We say that $\mathbf{T}$ is multivariate increasing failure rate relative to $(\Im_t)_{t\geq 0}$, denoted by MIFR$|\Im_t$, if for all $0 \leq t \leq t^*$ and all open upper sets $U \in \Re^n$

$$P(\theta_t \mathbf{T} \in U|\Im_t) \geq P(\theta_t \mathbf{T} \in U|\Im_{t^*}),$$

where a set $U \subset \Re^n$ is an upper set if, $\mathbf{x} \in U$ and $\mathbf{y} \geq \mathbf{x}$ implies $\mathbf{y} \in U$.

Remark 2.2.2 The MIFR$|\Im_t$ distribution class have most of what could be called "desirable properties" of any extension of the conventional IFR class. It is of some interest to see whether the σ-algebra $\Im_t$ can be changed into some smaller σ-algebra $G_t \subset \Im_t$ for the purpose of dealing with the subvector $\mathbf{T}_0$. Note that in general, if $G_t \subset \Im_t$, neither of the implications
$\mathbf{T}$ is MIFR$|\Im_t \Rightarrow \mathbf{T}$ is MIFR$|G_t$ or $\mathbf{T}$ is MIFR$|G_t \Rightarrow \mathbf{T}$ is MIFR$|\Im_t$ needs to be true.

Definition 2.2.3 We say that $\mathbf{T}$ is multivariate new better than used relative to $(\Im_t)_{t\geq 0}$, denoted by MNBU$|\Im_t$, if

$$P\{\theta_t \mathbf{T} \in U|\Im_t\} \leq P\{\mathbf{T} \in U|\Im_0\}.$$

The class of MIFR$|\Im_t$ distributions is contained in the class of MNBU$|\Im_t$ distributions. Moreover the class MNBU$|\Im_t$ has a closure property which is not enjoyed by the MIFR$|\Im_t$ class: the σ-algebras can be made coarser without destroying the MNBU$|\Im_t$ property , that is

Lemma 2.2.4 Suppose that $\mathbf{T}$ is MNBU$|\Im_t$. Let $I_0 \subset C$ a subset index of components and

$$G_t = \sigma\{1_{\{T_{(i)}>s\}}, X_i = j, i \in I_0, j \in I_0, 0 < s \leq t\}.$$

Then $\mathbf{T}_0 = (T_i)_{i\in I_0}$ is MNBU$|G_t$.

3. Section 3
3. Preservation results

In the next theorem we discuss the preservation of the component lifetimes MIFR$|\Im_t$ property under the point process signature representation. In fact, the MIFR$|\Im_t$ is carried over the system lifetime. **Theorem**

3.1 Let $\mathbf{T} = (T_1, ..., T_n)$ be the component lifetimes of a coherent system with lifetime T. If $\mathbf{T}$ is MIFR$|\Im_t$, then T is IFR$|\Im_t$. **Proof** As $T_{(k)j}, 1 \leq k \leq n, 1 \leq j \leq n$ are increasing functions of $\mathbf{T}$, it can be proved that

$T_{(k)j}$ is IFR$|\Im_t$. Therefore, if $t^* > t$ we have, for all $\Delta \in \Im_t$

$$P(\theta_t T_{(k)j} > s, \Delta) = \int_\Delta P(\theta_t T_{(k)j} > s|\Im_t)dP \geq$$

$$\int_\Delta P(\theta_{t^*} T_{(k)j} > s|\Im_{t^*})dP = P(\theta_{t^*} T_{(k)j} > s, \Delta).$$

However $\{T = T_{(k)j}\} \in \Im_{T_{(k)j}}$, where

$$\Im_{T_{(k)j}} = \{A \in \Im_\infty : A \cap \{T_{(k)j} > t\} \in \Im_t, \forall t\}$$

and we conclude that $\{T = T_{(k)j}\} \cap \{T_{(k)j} > t\} \in \Im_t$.
Replacing $\Delta = \{T = T_{(k)j}\} \cap \{T_{(k)j} > t\} \in \Im_t$ we have

$$P(T_{(k)j} > t+s, T_{(k)j} = T) = P(T_{(k)j} > t+s, T_{(k)j} = T, T_{(k)j} > t) \geq$$

$$P(T_{(k)j} > t^*+s, T_{(k)j} = T, T_{(k)j} > t^*) = P(T_{(k)j} > t^*+s, T_{(k)j} = T).$$

Therefore

$$P(\theta_t T > s|\Im_t) = P(T > t+s|\Im_t) = E[P(T > t+s|\Im_{t+s})|\Im_t] =$$

$$E[\sum_{j,k=1}^n 1_{\{T=T_{(k)j}\}} 1_{\{T_{(k)j} > t+s\}}|\Im_t] \geq E[\sum_{j,k=1}^n 1_{\{T=T_{(k)j}\}} 1_{\{T_{(k)j} > t^*+s\}}|\Im_{t^*}] =$$

$$E[P(T > t^*+s|\Im_{t^*+s})|\Im_{t^*}] = P(\theta_{t^*} T > s|\Im_{t^*}).$$

Remark 3.2 This result contradicts the well knows result that monotone system with independent IFR component lifetimes need not be IFR.

Similarly, as in the MIFR$|\Im_t$ distributions class, the point process signature representation preserves the MNBU$|\Im_t$ property.

Theorem 3.3 Let $\mathbf{T} = (T_1, ..., T_n)$ be the component lifetimes of a coherent system with lifetime T. If $\mathbf{T}$ is MNBU$|\Im_t$, then T is NBU$|\Im_t$. **Proof** As $T_{(k)j}, 1 \leq k \leq n, 1 \leq j \leq n$ are increasing functions of $\mathbf{T}$, it can be proved that $T_{(k)j}$ is NBU$|\Im_t$ and

$$P(\theta_t T_{(k)j} > s|\Im_t) \leq P(T_{(k)j} > s|\Im_0), \quad 1 \leq k \leq n, 1 \leq j \leq n.$$

Therefore

$$P(\theta_t T > s|\Im_t) = E[E\{1_{\{T>t+s\}}|\Im_{t+s})|\Im_t] =$$

$$\sum_{j,k=1}^n E[1_{\{T_{(k)j}=T\}} 1_{\{T_{(k)j} > t+s\}}|\Im_t] =$$

$$\sum_{j,k=1}^n 1_{\{T_{(k)j}=T\}} E[1_{\{T_{(k)j} > t+s\}}|\Im_t] \leq \sum_{j,k=1}^n 1_{\{T_{(k)j}=T\}} E[1_{\{T_{(k)j} > s\}}|\Im_0] \leq$$

$$E\{\sum_{j,k=1}^n 1_{\{T_{(k)j}=T\}} 1_{\{T_{(k)j} > s\}}|\Im_0] = P(T > s|\Im_0).$$

Unlike the class of MIFR$|\Im_t$ distributions, the class MNBU$|\Im_t$ has a closure property which is not enjoyed by the MIFR$|\Im_t$ class: the σ-algebras can be made coarser without destroying the MNBU$|\Im_t$ property (see Remark 3.2). Therefore we can prove that:

Theorem 3.4 Let $\mathbf{T} = (T_1, ..., T_n)$ be the component lifetimes of a coherent system with lifetime T. If $\mathbf{T}$ is MNBU$|\Im_t$, $(G_t)_{t\geq 0}$ is a subσalgebra of $(\Im_t)_{t\geq 0}$, that is, $G_t \subseteq \Im_t, \ \ \forall t \geq 0$ and $G_0 = \Im_0$, then T is NBU$|G_t$.
Proof

$$P(\theta_t T > s|G_t) = E[P(\theta_t T > s|\Im_t)|G_t] \leq$$
$$E[P(T > s|\Im_0)|G_t] = E[P(T > s|G_0)|G_t] = P(T > s|G_0).$$

4. Section 4

4. Inspecting the system at a fixed time t

As in Samaniego et al.(2009), in time dynamics we can observe the event $\{T > t\} \cap \{T_{(i)} \leq t \leq T_{(i+1)}\}$. However it is well know that

$$\Im_t \cap \{T_{(i)} \leq t \leq T_{(i+1)}\} = \Im_{T_{(i)}} \cap \{T_{(i)} \leq t \leq T_{(i+1)}\},$$

that is, the information up to t is the same information up to $T_{(i)}$. It means that, after the i-th failure we continue to observe $(\Im_{T_{(i)}+t})_{t\geq 0}$, where

$$\Im_{T_{(i)}+t} = \{A \in \Im_\infty : A \cap \{T_{(i)} \leq s - t\} \in \Im_s, \forall s > 0\}.$$

Using the Randon Sample Theorem we can say that for $0 \leq t \leq t^*$ and any open upper set $U \in \Re^n$, we have

$$P(\theta_t(\mathbf{T} - T_{(i)})^+ \in U|\Im_{T_{(i)}+t}) \geq P(\theta_{t^*}(\mathbf{T} - T_{(i)})^+ \in U|\Im_{T_{(i)}+t^*}),$$
$$P(\theta_t(\mathbf{T} - T_{(i)})^+ \in U|\Im_{T_{(i)}+t}) \geq P((\mathbf{T} - T_{(i)})^+ \in U|\Im_{T_{(i)}}),$$

preserving the MIFR$|\Im_t$ and the MNBU$|\Im_t$ classes of distributions.
Also and conveniently, after the i-th failure, we can consider the signature process

$$(P(T = T_{(i+1)j_{i+1}}|\Im_{T_{(i)}}), P(T = T_{(i+2)j_{i+2}}|\Im_{T_{(i)}}), ..., P(T = T_{(n)j_n}|\Im_{T_{(i)}}))$$

and the representation

$$P(T > t|\Im_{T_{(i)}+t}) = \sum_{k,j_k=i+1}^{n} P(T = T_{(k)j_k}|\Im_{T_{(i)}})P(T_{(k)j_k} > t|\Im_{T_{(i)}}).$$

5. Conclusions The classical signature coherent system lifetime distribution representation give us the motivation to represent, in general, the lifetime distribution of a coherent system through a combination of the distribution functions of the ordered components lifetimes. It is done through a point process approach. Given the results, it is mandatory to review the classical reliability properties in this new context. We are concerning in determine, for ach class of lifetime distribution, whether the formation of a coherent systems yields a life distribution within the same class. It is well know that, if each component lifetime distribution of a coherent system is NBU, then the system has an NBU lifetime distribution. Otherwise, a coherent system of independent IFR components lifetimes, need not have IFR lifetime distribution. Conveniently, working with signature point processes, we use the Arjas extension concepts of MIFR$|\Im_t$ and NBU$|\Im_t$ to prove that theses lasses are preserved under the formation of coherent systems.

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