Parametric estimation for fractional stochastic differential equations with the `Yuima` package

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Abstract

The fractional Brownian motion is a useful generalization of the Wiener process. It presents the long range dependence property which is observed in several applications in finance, biology, physics, internet traffic, *etc.*. Its paths can be more regular (in the Hölder sense) than the Wiener paths. But solutions of stochastic differential equations driven by fractional Brownian motion are no longer semimartingale and do not present the Markov property. In this presentation, we will review the estimation procedures that still work in this context (for the high-frequency scheme, for the large sample convergence scheme and for the mixed scheme). The long range dependence property of the solutions implies generally to deal with all the correlations of the sample components to reach efficiency. It leads to heavier computational estimators. If we seek rate-optimality, faster algorithms can be presented for some values of the Hurst parameter. Our theoretical results will be illustrated with the `Yuima` package.

Keywords: fractional Brownian motion; high-frequency scheme; generalized quadratic variations; maximum of contrast estimator.

1. Introduction

Parametric estimation in diffusion processes has been intensively studied for the three last decades. For $T > 0$, $H > \frac{1}{2}$ and Θ an open subset of $\mathbb{R}^p$, it is possible to define, with the generalization of the Riemann-Stieljes integral, the solution $Y^T = (Y_t, 0 \leq t \leq T)$ of the stochastic differential equation

$$Y_t = y_0 + \int_0^t V_0(Y_s, \vartheta) ds + \int_0^t V_1(Y_s, \vartheta) dW_s^H, \quad 0 \leq t \leq T, \tag{1}$$

where $\left(W_t^H, 0 \leq t \leq T\right)$ is the standard fractional Brownian motion (fBm), *i.e.* the centered Gaussian process of covariance function

$$\mathbf{E} W_s^H W_t^H = \frac{1}{2}\left(s^{2H} + t^{2H} - (t-s)^{2H}\right), \quad 0 < s < t,$$

and $\vartheta \in \Theta$ is the parameter to be estimated.
In practice, we observe the path of process on a temporal discrete grid $0 = t_0^n < t_1^n < \ldots < t_n^n, \quad n \geq 1$. In the following we consider in the following regular grids for which the mesh size is $\Delta_n = t_1^n - t_0^n = t_2^n - t_1^n = \ldots = t_n^n - t_{n-1}^n$. The observation sample is denoted

$$Y^{(n)} = (Y_{t_1}, \ldots, Y_{t_n}).$$

Asymptotical properties of the estimators of ϑ depends on the convergence scheme. Classically, there are three main convergence scheme:

1. the "high-frequency" scheme where the observation horizon $t_n^n = T > 0$ is fixed and the mesh size $\Delta_n = \frac{T}{n} \longrightarrow 0$ when the number of observation n tends to infinity.
2. the "large sample" scheme where the mesh size $\Delta_n = \Delta > 0$ is fixed and the observation horizon $t_n^n = n\Delta \longrightarrow \infty$ when the number of observation n tends to infinity.
3. the "mixed scheme" where the observation horizon $t_n^n = n\Delta_n \longrightarrow \infty$ and the mesh size $\Delta_n \longrightarrow 0$ simultaneously.

2. Inference for diffusion processes

For $H = \frac{1}{2}$, the standard fBm is the Wiener process and the equation (1) is defined in the Itò sense. Consequently, for $\vartheta \in \Theta$, the solution $(Y_t,\ 0 \leq t \leq T)$ of equation (1) is a homogenous Markov process whose transition probability densities are denoted $p^{\vartheta}(t, x, y)$. Consequently, the likelihood function can be written as

$$\mathcal{L}(\vartheta, Y^{(n)}) = \prod_{i=1}^{n} p^{\vartheta}(\Delta_n, Y_{t_{i-1}}, Y_{t_i}).$$

For some diffusion processes, the transition probability densities are in closed form and exact maximum likelihood

$$\widehat{\vartheta}_n = \arg\max_{\vartheta \in \Theta} \mathcal{L}(\vartheta, Y^{(n)}).$$

can be performed (Ornstein-Uhlenbeck process, geometric brownian motion, Cox-Ingersoll-Ross process) and its asymptotical properties can be studied. Otherwise, likelihood function has no explicit form. But transition densities can be approximated by Gaussian densities and the quasi-likelihood can be defined both with maximum of quasi-likelihood estimator.

In the high-frequency scheme, the LAMN (*local asymptotic mixing normality*) property of the likelihoods has been established in [4]. When it is satisfied, minimax theorems of [8, 9] are valid and lower bounds for the estimator variance can be deduced. Moreover, efficient estimation method are given in [4, 5] based on the quasi-likelihood.

In the large sample scheme, LAN (local symptomatic normality) property of the likelihoods has been established in the ergodic and uniformly elliptic setting [15]. Moreover, efficient estimators can be constructed by approximating the likelihood : numerical approximation of the Fokker-Planck-Kolmogorov, Monte-Carlo simulation, Hermite expansions and other methods.

Finally, for the mixed scheme LAN property of the likelihoods has been established in the ergodic and uniformly elliptic setting by [6]. Efficient estimators can be found in [10] for instance. They suppose that $n\Delta_n^p \longrightarrow 0$ for some $p > 2$.

3. Inference for fractional stochastic differential equations

For $H > \frac{1}{2}$, the fBm is neither Markov, nor martingale and its increments are no more independent:

$$\begin{aligned} r(s) = \mathbf{E}W_1^H(W_{s+1}^H - W_s^H) &= \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left(\prod_{k=0}^{2n-1} (2H - k) \right) s^{2H-2n} \\ &= H(2H-1)s^{2H-2} + \underset{s \to \infty}{\mathcal{O}}(s^{2H-4}). \end{aligned}$$

The correlation is positive and

$$\lim_{N \to \infty} \sum_{n=0}^{N} r(n) = \infty$$

which is known as long range dependance. Consecutively, solutions of (1) are also neither Markov nor semimartingale and classical tools do no work anymore.

In the high-frequency scheme, LAMN property of the likelihoods has been established in [11] for stochastic differential equations (1) with no drift and uniformly elliptic diffusion coefficient for a known Hurst exponent. Rate-efficient estimators based on contrasts are also described. We present in our talk a procedure, adapted from [1] and [11], for the joint estimation of the parameter in the diffusion coefficient and Hurst exponent which has been added to `Yuima`. In this setting, goodness-of-fit tests has also been developed [14].

In the two other schemes, such general results are difficult to obtain. Some works has been done for the fractional Ornstein-Uhlenbeck process (fOUp), solution of the stochastic differential equation

$$Y_t = y_0 - \lambda \int_0^t Y_s ds + \sigma W_t^H, \quad t \geq 0, \tag{2}$$

The solution, given by

$$Y_t = e^{-\lambda t}\left(y_0 + \sigma \int_0^t e^{\lambda s} dW_s^H\right), \quad t \geq 0,$$

is Gaussian and ergodic for $\lambda > 0$. For $H > \frac{1}{2}$ it present long range dependance with

$$\begin{aligned} r(s) = \mathbf{E}Y_0 Y_s &= c_H \int_{-\infty}^{\infty} e^{isx} \frac{|x|^{1-2H}}{\lambda^2 + x^2} dx \\ &= \sigma^2 H(2H-1)\lambda^{-2} s^{2H-2} + \underset{s\to\infty}{\mathcal{O}}(s^{2H-4}). \end{aligned} \tag{3}$$

In the large sample scheme, exact MLE and Whittle estimators has been studied [7]. Asymptotic efficiency in Fisher sense has been obtained. We will show, using [3], that in this Gaussian case, the LAN property of the likelihoods holds.
Finally, in the mixed scheme, different estimating procedures has been studied when the Hurst paper is known : moment estimation for λ, least-square estimation for λ in [13] or with integral transformation. A joint estimation procedure for parameters and H has been developed in [2].

4. The `Yuima` package

The `yuima` package is a comprehensive framework, based on the S4 system of classes and methods, which allows for the description of solutions of stochastic differential equations. Although we can only give few details here, the user can specify a stochastic differential equation implying Levy process and fBm (recall that for $H = \frac{1}{2}$ the fBm is the standard Brownian motion).
Assume first that we want to describe the following fOUp

$$Y_t = 1 - 2\int_0^t Y_t dt + dW_t^H, \qquad H = 0.7.$$

This is done in `yuima` specifying the drift and diffusion coefficients as plain mathematical expressions

```
R> mod <- setModel(drift="-2*x", diffusion="1", hurst=0.7)
```

With this specification, using the following command

```
R> str(mod)
```

it is possible to see that the jump coefficient is void and the Hurst parameter is set to 0.7. Now, with `mod` in hands, it is very easy to simulate a trajectory of the process (on the default grid) and eventually plot it on screen as follows

```
R> set.seed(123)
R> X <- simulate(mod)
R> plot(X)
```

On a regular grid, the fractional Gaussian noise is simulated with the Wood and Chan method [16]. Sample path are simulated with the Euler-Maruyama method [12].
For specifying parametric statistical model, for instance the geometric fractional Brownian motion (gfBm)

$$Y_t = 1 + \theta_1 \int_0^t Y_s ds + \theta_2 \int_0^t Y_s dW_s^H,$$

the `yuima` package attempts to distinguish the parameters' names from the ones of the state and time variables. Here, the user can follow

```
R> mod2 <- setModel(drift = "theta1*x", diffusion = "theta2*x",hurst="NA")
R> str(mod2)
```

in order to understand how parametric models are handled internally.
The `yuima` package provides the user, not only the simulation part, but also several parametric and non-parametric estimation procedures. During the talk, we show how to use `yuima` for simulation of previously mentioned fractional stochastic differential equations (fOUp, gfBm,...) and we present different statistical methods for parameter estimation. For instance, the estimation procedure for the Hurst exponent based on generalized quadratic variations has been implemented in `qgv` function that works for a large class of fractional diffusions. The procedure for joint estimation in the high-frequency scheme based on contrast is called `mcontrast`. The procedure for joint estimation of the Hurst exponent H, diffusion coefficient σ and drift parameter λ in the mixed scheme is called `lse(,frac=TRUE)`.

References

[1] C. Berzin and J. Leon. Estimation in models driven by fractional Brownian motion. *Annales de l'Institut Henri Poincaré*, 44(2):191–213, 2008.

[2] A. Brouste and S. Iacus. Parameter estimation for the discretely observed fractional Ornstein-Uhlenbeck process and the yuima r package. *Computational Statistics*, 28(4):1529–1447, 2013.

[3] S. Cohen, F. Gamboa, C. Lacaux, and J.-M. Loubes. LAN property for some fractional type Brownian motion. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 10(1):91–106, 2013.

[4] G. Donhal. On estimating the diffusion coefficient. *Journal of Applied Probability*, 24:105–114, 1987.

[5] V. Genon-Catalot and J. Jacod. On the estimation of the diffusion coefficient for multi-dimensional diffusion processes. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 29(1):119–151, 1993.

[6] E. Gobet. LAN property for ergodic diffusions with discrete observations. *Annales de l'Institut Henri Poincaré*, 38(5):711–737, 2002.

[7] H. Hult. *Topics on fractional Brownian motion and regular variation for stochastic processes*. PhD thesis, Stockholm Royal Institute of Technology, 2003.

[8] P. Jeganathan. On the asymptotic theory of estimation when the limit of the log-likelihood ratios is mixed normal. *Sankhya*, 44(2):173–212, 1982.

[9] P. Jeganathan. Some asymptotic properties of risk functions when the limit of the experiment is mixed normal. *Sankhya*, 45(1):66–87, 1983.

[10] M. Kessler. Estimation of an ergodic diffusion from discrete observations. *Scandinavian Journal of Statistics*, 24:211–229, 1997.

[11] C. Ludena. Minimum contrast estimation for fractional diffusion. *Scandinavian Journal of Statistics*, 31:613–628, 2004.

[12] A. Neuenkirch and I. Nourdin. Exact rate of convergence of some approximation schemes associated to SDEs driven by a fractional Brownian motion. *J. Theoret. Prob.*, 20(4):871–899, 2007.

[13] A. Neuenkirch and S. Tindel. A Least Square-type procedure for parameter estimation in stochastic differential equations with additive fractional noise. *Statistical Inference for Stochastic Processes*, 17(1):99–120, 2014.

[14] M. Podolskij and K. Wasmuth. Goodness-of-fit testing for fractional diffusions. *Statistical Inference for Stochastic Processes*, 16(2):147–159, 2013.

[15] G. Roussas. *Contiguity of probability measures*. Cambridge University Press, 1972.

[16] A. Wood and G. Chan. Simulation of stationary Gaussian processes in $[0,1]^d$. *Journal of computational and graphical statistics*, 3(4):409–432, 1994.