Mixed Poisson Process with Pareto Mixing variable - Revisited

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Abstract

In 2013 Jodanova, Dusek and Stehlik consider Mixed Poisson Process with Pareto mixing variable and introduce Mixed Poisson-Pareto, Exponential-Pareto and Erlang-Pareto distributions. The aim of this paper is to popularize these results. Some new results are given: invariance of Exponential-Pareto type with respect to integrated tail transformations, the distribution of the tinning variable of the Mixed Poisson-Pareto distribution and description of the tinned Mixed Poisson-Pareto process.

Keywords: Generalized exponential integral function; Gamma-Pareto distribution; Integrated tail distribution; Tinning variable.

1. Introduction

Mixed Poisson processes (MPPs) are considered e.g. in Willmot (1993), J. Grandell (1997) and Karlis et al. (2005). The most popular MPP is the one with Gamma mixing variable. Its inter-arrival times are Pareto distributed and the number of "events" up to time t is Negative binomial. See Greenwood and Yule (1913). Jordanova et al.(2013) investigate Mixed Poisson process with Pareto mixing variable (MPPP). They define the Exponential-Pareto distribution (EPD) as the distribution of the inter-arrival times of MPPP and generalise it to Gamma - Pareto distribution. The last distribution describes the moment of the "n-th event". The Poisson-Pareto distribution counts the the number of "events" up to time t. Here these results are revisited and some new properties are given.
The paper is organised as follows. In the second section we remind the definition for EPD and prove that this type is invariant with respect to integrated tail transformations (this transformation increases only the first parameter). In Second 3 the p-thinning distribution of Mixed Poisson-Pareto distribution is found. Section 4 contains description of MPPP. We show that its thinning is again MPPP.
Along the paper $\Lambda_{\alpha,\delta}$ is a Pareto distributed random variable (r.v.) with parameters $\alpha > 0$ and $\delta > 0$ (briefly $\Lambda_{\alpha,\delta} \sim Pareto(\alpha, \delta)$). Its cumulative distribution function (c.d.f.) is

$$F_{\Lambda_{\alpha,\delta}}(x) = \begin{cases} 0 & , \quad x \leq \delta \\ 1 - \frac{\delta^\alpha}{x^\alpha} & , \quad x > \delta \end{cases} .$$

2. The Exponential-Pareto distribution

Definition 1. A r.v. $\tau_{\alpha,\delta}$ has Exponential-Pareto distribution with parameters $\alpha > 0$, $\delta > 0$ if its c.d.f. is

$$F_{\tau_{\alpha,\delta}}(x) = \begin{cases} 0 & , \quad x \leq 0 \\ 1 - \alpha E_{\alpha+1}(\delta x) & , \quad x > 0 \end{cases} , \tag{1}$$

where

$$E_p(z) = z^{p-1}\Gamma(1-p, z), \quad p \in R, \quad z > 0, \quad \Gamma(x, t) = \int_t^\infty y^{x-1} e^{-y} dy, \quad p \in R, \quad t > 0$$

are correspondingly the generalized exponential integral and the upper incomplete Gamma function. Briefly $\tau_{\alpha,\delta} \sim EP(\alpha, \delta)$. It is a proper distribution therefore the renewal process born by it is not terminating.
More information about these functions could be found e.g. in Olver et al. (2010) or Milgram (1985). Plots of these c.d.fs could be seen by Wolfram alpha. E.g. for $\alpha = 1$ and $\delta = 1$ you could just type $plot[1 - ExpIntegralE[2, x], \{x, 0, 2\}]$ in its command line.
The probability density function (p.d.f.) is

$$P_{\tau_{\alpha,\delta}}(x) = \alpha\delta E_\alpha(\delta x), x > 0, \tag{2}$$

its Laplace transform is

$$Ee^{-t\tau_{\alpha,\delta}} = \alpha\delta^\alpha t^{-\alpha}\int_{\delta/t}^\infty z^{-\alpha}(1+z)^{-1}dz = \alpha\delta\int_0^\infty e^{-ty}E_\alpha(\delta y)dy,$$

and its moments and variance are finite

$$E\tau_{\alpha,\delta}^k = \frac{\alpha k!}{\delta^k(k+\alpha)},\ k=1,2,...,Var\tau_{\alpha,\delta} = \frac{\alpha(2+2\alpha+\alpha^2)}{\delta^2(\alpha+1)^2(\alpha+2)}. \tag{3}$$

This distribution arises as exponential distribution with Pareto mixing variable and possesses the following scale property $\tau_{\alpha,\delta} \stackrel{d}{=} \delta\tau_{\alpha,1}$, where $\stackrel{d}{=}$ means equality in distribution. It coincides with the one of the fraction of independent standard exponential E_1 and Pareto distributed random variables $\Lambda_{\alpha,\delta}$. More precisely

$$\tau_{\alpha,\delta} \stackrel{d}{=} \frac{E_1}{\delta\Lambda_{\alpha,1}}. \tag{4}$$

In the next proposition we find the failure rate of this distribution and prove that its integrated tail transformation only increases the first parameter of the distribution, but keeps the same type.
Theorem 1. Let $\tau_{\alpha,\delta} \sim EP(\alpha,\delta)$ and $F_I(x) = \frac{1}{E\tau_{\alpha,\delta}}\int_0^x(1-F_{\tau_{\alpha,\delta}}(y))dy$ the integrated tail distribution. Then

$$s\tau_{\alpha,\delta} \stackrel{d}{=} \tau_{\alpha,\frac{\delta}{s}},\, s>0. \tag{5}$$

The failure rate is

$$\frac{P_{\tau_{\alpha,\delta}}(x)}{1-F_{\tau_{\alpha,\delta}}(x)} = \frac{e^{-\delta x}}{xE_{\alpha+1}(\delta x)} - \frac{\alpha}{x} \tag{6}$$

and the integrated tail distribution is $EP(\alpha+1,\delta)$.
Proof. (5) is an immediate consequence of (4). (6) follows after replacing (1) and (2) in the definition for the failure rate. In order to prove (6) consider $x>0$.

$$F_I(x) = \frac{\delta(\alpha+1)}{\alpha}\int_0^x \alpha E_{\alpha+1}(\delta y)dy = \delta(\alpha+1)\int_0^x E_{\alpha+1}(\delta y)dy = (\alpha+1)\int_0^x E_{\alpha+1}(\delta y)d(\delta y) =$$

$$= (\alpha+1)[E_{\alpha+2}(0) - E_{\alpha+2}(\delta x)] = (\alpha+1)[\frac{1}{\alpha+1} - E_{\alpha+2}(\delta x)] = 1-(\alpha+1)E_{\alpha+2}(\delta x) = P(\tau_{\alpha+1,\delta} < x).$$

Definition 2. A random vector $(\tau_{\alpha,\delta,1},\tau_{\alpha,\delta,2},...,\tau_{\alpha,\delta,n})$ with

$$P(\tau_{\alpha,\delta,1}\geq x_1,\tau_{\alpha,\delta,2}\geq x_2,...,\tau_{\alpha,\delta,n}\geq x_n) = \alpha.E_{\alpha+1}(\delta(x_1+...+x_n)),\quad x_i\geq 0,\ i=1,2,...,n,$$

is called Exponentially-Pareto distributed random vector. Briefly $(\tau_{\alpha,\delta,1},\tau_{\alpha,\delta,2},...,\tau_{\alpha,\delta,n}) \sim \mathbf{EP}(\alpha,\delta)$.
Any subset of its coordinates is again $\mathbf{EP}(\alpha,\delta)$ distributed and its copula is Archimedian survival copula with generating function $\phi(t)$ such that

$$\phi^{\leftarrow}(t) = \alpha\delta\int_0^\infty e^{-t.y}E_\alpha(y\delta)dy.$$

Here $\phi^{\leftarrow}(t)$ means the inverse function of $\phi(t)$.
There exist a probability space, independent identically distributed (i.i.d.) standard exponential r.vs $E_1,E_2,...,E_n$ and independent on them r.v. $\Lambda_{\alpha,\delta}\sim Pareto(\alpha,\delta)$ on the same probability space such that

$$(\tau_{\alpha,\delta,1},\tau_{\alpha,\delta,2},...,\tau_{\alpha,\delta,n}) \stackrel{d}{=} \left(E_1\Lambda_{\alpha,\delta}^{-1},E_2\Lambda_{\alpha,\delta}^{-1},...,E_n\Lambda_{\alpha,\delta}^{-1}\right).$$

If $\Lambda_{\alpha,\delta}\sim Pareto(\alpha,\delta)$ and $(\tau_1,\tau_2,...,\tau_n)$ are defined by

$$P(\tau_1\geq x_1,\tau_2\geq x_2,...,\tau_n\geq x_n) = \int_\delta^\infty e^{-y.(x_1+...+x_n)}P_{\Lambda_{\alpha,\delta}}(y)dy,\ x_i\geq 0,\ i=1,2,...,n,$$

then $(\tau_{\alpha,\delta,1}, \tau_{\alpha,\delta,2}, ..., \tau_{\alpha,\delta,n}) \stackrel{d}{=} (\tau_1, \tau_2, ..., \tau_n)$.

Definition 3. A r.v. $T_{\alpha,\delta,n}$ with survival function

$$P(T_{\alpha,\delta,n} \geq x) = \alpha(\delta x)^\alpha \sum_{i=0}^{n-1} \frac{\Gamma(i-\alpha, \delta x)}{i!}, \quad x \geq 0 \tag{7}$$

is called **Erlang-Pareto** distributed with parameters $\alpha > 0$, $\delta > 0$ and $n \in \mathbf{N}$. Briefly $T_{\alpha,\delta,n} \sim ErlP(\alpha, \delta, n)$.

$$P_{T_{\alpha,\delta,n}}(x) = \alpha\delta\frac{(\delta x)^{\alpha-1}}{(n-1)!}\Gamma(n-\alpha, \delta x) = \frac{\alpha\delta^n x^{n-1}}{(n-1)!}E_{\alpha-n+1}(\delta x), \quad x \geq 0.$$

$$ET^k_{\alpha,\delta,n} = \frac{\alpha(n+k-1)!}{(n-1)!\delta^k(\alpha+k)}, \quad k = 1, 2, ..., \quad VarT_{\alpha,\delta,n} = \frac{\alpha n(n+(\alpha+1)^2)}{\delta^2(\alpha+2)(\alpha+1)^2}.$$

$$Ee^{-tT_{\alpha,\delta,n}} = \frac{\alpha\delta^\alpha}{t^\alpha}\int_{\delta/t}^\infty \frac{z^{n-\alpha-1}}{(1+z)^n}dz.$$

Let $T_{1,n}$ be an Erlang distributed r.v. with parameters 1 and n independent on $\Lambda_{\alpha,\delta} \sim Pareto(\alpha, \delta)$, then

$$T_{\alpha,\delta,n} \stackrel{d}{=} \frac{T_{1,n}}{\Lambda_{\alpha,\delta}} \stackrel{d}{=} \frac{T_{1,n}}{\delta\Lambda_{\alpha,1}} \stackrel{d}{=} \frac{T_{\alpha,1,n}}{\delta}.$$

In analogous way we could proceed with the following definition.

Definition 4: We say that the random variable $\nu_{a,\alpha,\delta}$ is Gamma-Pareto distributed with parameters $a > 0$, $\alpha > 0$ and $\delta > 0$, briefly $\nu_{a,\alpha,\delta} \sim G - P(a, \alpha, \delta)$, if it has the following p.d.f.

$$P_{\nu_{a,\alpha,\delta}}(x) = \frac{\alpha.\delta^\alpha x^{\alpha-1}}{\Gamma(a)}\Gamma(a-\alpha, x\delta) = \frac{\alpha.\delta^a x^{a-1}}{\Gamma(a)}E_{\alpha-a+1}(x\delta), \quad x > 0.$$

Obviously $ErlP(\alpha, \delta, n)$ distribution coincides with $G - P(n, \alpha, \delta)$ and if $\nu_{a,\alpha,\delta} \sim G - P(a, \alpha, \delta)$, then

$$\nu_{a,\alpha,\delta} \stackrel{d}{=} \frac{\gamma_{a,1}}{\Lambda_{\alpha,\delta}} \stackrel{d}{=} \frac{\gamma_{a,1}}{\delta.\Lambda_{\alpha,1}} \stackrel{d}{=} \frac{\nu_{a,\alpha,1}}{\delta},$$

where $\gamma_{a,1}$ is Gamma distributed random variable with parameters a and 1, independent on $\Lambda_{\alpha,\delta}$.

$$E\nu^k_{a,\alpha,\delta} = \frac{(a+k-1)...a\alpha}{(\alpha+k)\delta^k}, k = 1, 2, ..., \quad Var\,\nu_{a,\alpha,\delta} = \frac{\alpha a[(\alpha+1)^2+a]}{\delta^2(\alpha+2)(\alpha+1)^2}.$$

$$Ee^{-s.\nu_{a,\alpha,\delta}} = \frac{\alpha.\delta^\alpha.}{s^\alpha}\int_{\delta/s}^\infty \frac{z^{a-\alpha-1}}{(1+z)^a}dz.$$

Proposition 1. If $\nu_{a,\alpha,\delta} \sim G - P(a, \alpha, \delta)$, then $s\nu_{a,\alpha,\delta} \stackrel{d}{=} \nu_{a,\alpha,\frac{\delta}{s}}$, $s > 0$.

3. Mixed Poisson-Pareto distribution

Definition 5. A random variable with probability mass function (p.m.f.)

$$P(N_{\alpha,\delta} = k) = \frac{\alpha.\delta^\alpha}{k!}\Gamma(k-\alpha, \delta) = \frac{\alpha.\delta^k}{k!}E_{\alpha-k+1}(\delta) \quad k = 0, 1, 2, ... \tag{8}$$

is called Mixed Poisson-Pareto random variable with parameters $\alpha > 0$, $\delta > 0$. Briefly $N_{\alpha,\delta} \sim M_{PP}(\alpha, \delta)$. It is a proper distribution and describes the number of "events" up to time t in a Mixed Poisson-Pareto process. See Jordanova et al. (2013).

$$EN_{\alpha,\delta} = \frac{\alpha.\delta}{\alpha-1}, \quad \alpha > 1, \quad Var\ N_{\alpha,\delta} = \frac{\alpha.\delta}{\alpha-1} + \frac{\alpha.\delta^2}{\alpha-2} - \left(\frac{\alpha.\delta}{\alpha-1}\right)^2, \quad \alpha > 2,$$

$$Ez^{N_{\alpha,\delta}} = \alpha.E_{\alpha+1}(\delta.(1-z)) = P(N_{\alpha,\delta(1-z)} = 0), \quad z \in (0,1). \tag{9}$$

The factorial moments

$$EN_{\alpha,\delta}(N_{\alpha,\delta}-1)...(N_{\alpha,\delta}-k+1) = E\Lambda^k_{\alpha,\delta} = \frac{\alpha.\delta^k}{\alpha-k}, \quad k = 1,2,...,[\alpha].$$

For $\alpha > 2$ these distributions are over dispersed. The index of dispersion is

$$\frac{Var\ N_{\alpha,\delta}}{E\ N_{\alpha,\delta}} = 1 + \frac{\delta}{(\alpha-1)(\alpha-2)}.$$

If $N_{\alpha,\delta} \sim M_{PP}(\alpha,\delta)$, then there exists a probability space and two random variables $\Lambda_{\alpha,\delta}$ and $N_{\Lambda_{\alpha,\delta}}$ defined on it such that $\Lambda_{\alpha,\delta} \sim Pareto(\alpha,\delta)$,

$$P(N_{\Lambda_{\alpha,\delta}} = k | \Lambda_{\alpha,\delta} = y) = \frac{y^k}{k!}e^{-y},\ y > \delta,\ k = 0,1,...$$

and $N_{\alpha,\delta} \stackrel{d}{=} N_{\Lambda_{\alpha,\delta}}$. Moreover $(\Lambda_{\alpha,\delta}|N_{\Lambda_{\alpha,\delta}} = n) \stackrel{d}{=} (\xi|\xi \geq \delta), n = [\alpha]+1, [\alpha]+2, ...$ where $\xi \sim Gamma(n-\alpha, 1)$. The following recursive formula allows us to calculate consecutively the probabilities in p.m.f.

$$P(N_{\alpha,\delta} = k) = \frac{\alpha}{k}P(Po(\delta) = k-1) - \frac{\alpha-k+1}{k}P(N_{\alpha,\delta} = k-1).$$

Now we will proceed with obtaining another properties of this distribution. Mecke (1968) proved that a discrete r.v. N is a mixed Poisson distributed if and only if it can be obtained by independent p-tinning for every $p \in (0,1)$. The last means that there exists a probability space, a discrete r.v. η_p and i.i.d. Bernoulli r.vs $I_{A_1}, I_{A_2}, ...$ defined on it, independent on η_p and such that $p = P(A_1)$ satisfying

$$N \stackrel{d}{=} I_{A_1} + ... + I_{A_{\eta_p}} \stackrel{d}{=}: Bi(\eta_p, p).$$

Theorem 2. 1. Let $N_{\alpha,\delta} \sim M_{PP}(\alpha,\delta)$, then for all $p \in (0,1)$, $N_{\alpha,\delta}$ could be obtained as independent p-tinning of $N_{\alpha,\frac{\delta}{p}}$, i.e. $N_{\alpha,\delta} \stackrel{d}{=} Bi(N_{\alpha,\frac{\delta}{p}}, p)$.
2. Suppose $N_{\Lambda_{\alpha,\delta}} \sim M_{PP}(\alpha,\delta)$ with mixing variable $\Lambda_{\alpha,\delta}$ and I_B is independent on it Bernoulli r.v., then for all $p \in (0,1)$, $N_{I_B\Lambda_{\alpha,\delta}} \stackrel{d}{=} I_B N_{\Lambda_{\alpha,\delta}} \stackrel{d}{=} Bi(I_B N_{\alpha,\frac{\delta}{p}}, p) \stackrel{d}{=} Bi(N_{I_B\Lambda_{\alpha,\frac{\delta}{p}}}, p)$.
3. If $N_1, N_2, ...|\Lambda_{\alpha,\delta} = y$ are conditionally i.i.d. $Po(y)$ distributed, then $N_1(\Lambda_{\alpha,\delta}) + ... + N_s(\Lambda_{\alpha,\delta}) \stackrel{d}{=} N_{\alpha,\delta s}$.

Proof: 1. Let $p \in (0,1)$. Consider i.i.d. Bernoulli r.vs $I_{A_1}, I_{A_2}, ...$ such that $p = P(A_1)$ and independent on them $N_{\alpha,\frac{\delta}{p}} \sim M_{PP}(\alpha, \frac{\delta}{p})$. We use $g_\xi(z)$ for the probability generating function (p.g.f.) of the r.v. ξ. By the properties of the p.g.f. of the compounds, (9) and the form of p.g.f. of the Bernoulli r.vs we obtain

$$g_{I_{A_1}+I_{A_2}+...+I_{A_{N_{\alpha,\frac{\delta}{p}}}}} = g_{N_{\alpha,\frac{\delta}{p}}}(g_{I_{A_1}}(z)) = \alpha E_{\alpha+1}(\frac{\delta}{p}(1-g_{I_{A_1}}(z))) =$$

$$= \alpha E_{\alpha+1}(\frac{\delta}{p}(1-(1-p+pz))) = \alpha E_{\alpha+1}(\delta(1-z)) = g_{N_{\alpha,\delta}}(z).$$

Definition 6: We say that a random vector $(N_{\alpha,\delta_1}, ..., N_{\alpha,\delta_n})$ is Ordered Poisson-Pareto distributed with parameters with parameters $\alpha > 0$, $0 < \delta_1 < .... < \delta_n$ (briefly $(N_{\alpha,\delta_1}, ..., N_{\alpha,\delta_n}) \sim O_{PP}(\alpha, \delta_1, ..., \delta_n)$) if

$$P(N_{\alpha,\delta_1} = k_1, ..., N_{\alpha,\delta_n} = k_n) = \frac{\alpha.\delta_1^{k_1}.\delta_2^{k_2-k_1}...\delta_n^{k_n-k_{n-1}}(\delta_1+...+\delta_n)^{\alpha-k_n}}{k_1!(k_2-k_1)!...(k_n-k_{n-1})!}.\Gamma(k_n-\alpha, \delta_1+...+\delta_n) =$$

$$= \frac{\alpha.\delta_1^{k_1}.\delta_2^{k_2-k_1}...\delta_n^{k_n-k_{n-1}}}{k_1!(k_2-k_1)!...(k_n-k_{n-1})!}.E_{1-k_n+\alpha}(\delta_1+...+\delta_n), \quad k = 0,1,..., \quad k_1 \leq k_2 \leq ... \leq k_n$$

and zero otherwise.

Definition 7: We say that a random vector $(N_{\alpha,\delta_1}, ..., N_{\alpha,\delta_n})$ is Mixed Poisson-Pareto distributed with parameters $\alpha > 0$, $\delta_1 > 0$, ..., $\delta_n > 0$ (briefly $(N_{\alpha,\delta_1}, ..., N_{\alpha,\delta_n}) \sim \vec{M}_{PP}(\alpha, \delta_1, ..., \delta_n)$) if

$$P(N_{\alpha,\delta_1} = k_1, ..., N_{\alpha,\delta_n} = k_n) = \frac{\alpha\delta_1^{k_1}\delta_2^{k_2}...\delta_n^{k_n}(\delta_1 + ... + \delta_n)^{\alpha-k_1-...-k_n}}{k_1!k_2!...k_n!}\Gamma(k_1 + ... + k_n - \alpha, \delta_1 + ... + \delta_n) =$$

$$= \frac{\alpha.\delta_1^{k_1}.\delta_2^{k_2}...\delta_n^{k_n}}{k_1!k_2!...k_n!}.E_{1-k_1-...-k_n+\alpha}(\delta_1 + ... + \delta_n), \quad k_1, k_2, ..., k_n \in \{0, 1, ...\}$$

and zero otherwise.

Proposition 2. If $(N_{\alpha,\delta_1}, ..., N_{\alpha,\delta_n}) \sim O_{PP}(\alpha, \delta_1, ..., \delta_n)$, then
$$(N_{\alpha,\delta_1}, N_{\alpha,\delta_2} - N_{\alpha,\delta_1} ..., N_{\alpha,\delta_n} - N_{\alpha,\delta_{n-1}}) \sim \vec{M}_{PP}(\alpha, \delta_1, \delta_2, ..., \delta_n).$$

Proposition 3. If $(N_{\alpha,\delta_1}, ..., N_{\alpha,\delta_n}) \sim \vec{M}_{PP}(\alpha, \delta_1, ..., \delta_n)$ then
$$(N_{\alpha,\delta_1}, N_{\alpha,\delta_1} + N_{\alpha,\delta_2} ..., N_{\alpha,\delta_1} + ... + N_{\alpha,\delta_n}) \sim O_{PP}(\alpha, \delta_1, \delta_2, ..., \delta_n).$$

4. Mixed Poisson-Pareto process

Let $\alpha > 0$, $\delta > 0$ and $\mathcal{A}$ be a sigma algebra with right-continuous filtration. Consider a probability space $\mathbf{\Omega} = (\Omega, \mathcal{A}, \mathbf{P})$ and a sequence $\tau_1, \tau_2, ...$ of positive r.vs defined on it such that for all $n \in N$, $(\tau_1, \tau_2, ..., \tau_n) \sim \mathbf{EP}(\alpha, \delta)$. Then for all $i = 1, 2, ...$ $\tau_i \sim EP(\alpha, \delta)$. We refer to the r.v. τ_i as the "i-th holding (waiting, inter-arrival) time". By (4) we could explain the dependence of τ_i by the parameters α and δ. The bigger the values of δ the shorter the periods between the consecutive "events". Decreasing of α entails heavier right tail of the c.d.f. of the mixing variable and more mass distributed by τ_i around zero, i.e. the consecutive events occur more frequently. Define
$$T_{\alpha,\delta,n} = \tau_{\alpha,\delta,1} + \tau_{\alpha,\delta,2} + ... + \tau_{\alpha,\delta,n}.$$
Jordanova et al. (2013) proved that $T_{\alpha,\delta,n} \sim ErlP(\alpha, \delta, n)$. This r.v. could be interpreted as "the time of n-th event". Denote the corresponding counting process by $N_{\alpha,\delta}$, i.e.

$$\{N_{\alpha,\delta}(t);\, t \geq 0\} = \{sup\{i \geq 0 : T_{\alpha,\delta,i} \leq t\},\, t \geq 0\}.$$

$N_{\alpha,\delta}(t)$ is the number of the "events" up to time t. Then $\{N_{\alpha,\delta}(t),\, t \geq 0\}$ is a Mixed Poisson Process with Pareto mixing variable. Let us remind the definition. See Jordanova et al. (2013).

Definition 8. Let N be a standard homogeneous Poisson process in $\mathbf{\Omega}$, $(EN(1) = 1)$ and $c(t)$ be a non-negative, strictly increasing continuous function, not obligatory starting from the coordinate beginning and $c(t) \to \infty$, $t \to \infty$. Assume $\Lambda_{\alpha,\delta}$ and N are independent. We call the random process

$$\{N_{\alpha,\delta}(t); t \geq 0\} = \{N(\Lambda_{\alpha,\delta}c(t)); t \geq 0\}$$

a **Mixed Poisson Process with Pareto mixing variable**. Briefly $\{N_{\alpha,\delta}(t); t \geq 0\} \sim MPPP(\alpha, \delta; c(t))$. As a particular case of MPPs, the MPPPs are Markov processes with dependent additive increments and finite number of jumps on any finite time interval. They are over-dispersed and posses the order statistics property. If $c(t) = t$ they are homogeneous in time. (See e.g. Mikosch, T. (2004)). Jordanova et al. (2013) prove that the distribution of the time intersections, finite dimensional distributions and the distribution of the additive increments of the MPPPs are as follows. If $\{N_{\delta,\alpha}(t); t \geq 0\} \sim MPPP(\alpha, \delta; c(t))$, $\alpha > 0$, $\delta > 0$, then for all $t > 0$, $N_{\alpha,\delta}(t) \sim M_{PP}(\alpha, \delta c(t))$. For $n \in \mathbf{N}$, $0 \leq t_1 < t_2 < ... < t_n$,

$$(N_{\alpha,\delta}(t_1), N_{\alpha,\delta}(t_2), ..., N_{\alpha,\delta}(t_n)) \sim O_{PP}(\alpha, \delta c(t_1), \delta(c(t_1) - c(t_1)), ..., \delta(c(t_n) - c(t_{n-1}))).$$

$$(N_{\alpha,\delta}(t_1), N_{\alpha,\delta}(t_2) - N_{\alpha,\delta}(t_1), ..., N_{\alpha,\delta}(t_n) - N_{\alpha,\delta}(t_{n-1})) \sim \vec{M}_{PP}(\alpha, \delta c(t_1), \delta(c(t_2) - c(t_1)), ..., \delta(c(t_n) - c(t_{n-1}))).$$

If $c(t) = t$, then $N_{\alpha,\delta}(t) \overset{f.d.d.}{=} N_{\alpha,1}(\delta t) \overset{d}{=} N_{\alpha,\delta t}(1)$. The last equality is true for all $t > 0$.

Denote by $\eta_{b,\alpha,\delta}(t) = t - T_{\alpha,\delta,N_{\alpha,\delta}(t)}$ - the length of the period $(T_{N_{\alpha,\delta}(t)}, t]$ since the last "event" occur, and by $\eta_{f,\alpha,\delta}(t) = T_{\alpha,\delta,N_{\alpha,\delta}(t)+1} - t$ - the length of the period $(t, T_{N_{\alpha,\delta}(t)+1}]$ until the next "event" occur. Then for all $t > 0$ $\eta_{f,\alpha,\delta}(t) \sim EP(\alpha, \delta)$ and $P(\eta_{b,\alpha,\delta}(t) \geq x_1) = \alpha E_{\alpha+1}(\delta x_1), x_1 \in [0, t]$. Due to the fact that the support of $\eta_{b,\alpha,\delta}(t)$ is bounded (it is $[0, t]$), the distribution of $\eta_{b,\alpha,\delta}(t)$ could be called "truncated EPD with parameters $\alpha > 0$ and $\delta > 0$". The common distribution of these r.vs is given by

$$P(\eta_{b,\alpha,\delta}(t) \geq x_1, \eta_{f,\alpha,\delta}(t) \geq x_2) = \alpha E\alpha + 1(\delta(x_1 + x_2)), x_1 \in [0, t], x_2 > 0.$$

Theorem 3. Consider i.i.d. r.vs $Y_1, Y_2, ...$ arriving according to $\{N_{\alpha,\delta}(t); t \geq 0\} \sim MPPP(\alpha, \delta; t)$. For any Borel set B,

$$\{N_{\alpha,\delta,p}(t); t \geq 0\} = \{\sum_{i=1}^{N_{\alpha,\delta}(t)} I\{Y_i \in B\}; t \geq 0\} \sim MPPP(\alpha, \frac{\delta}{p}; t), \quad P(Y_1 \in B) = p.$$

5. Conclusions

If we consider MPPP, Erlang-Pareto distribution characterizes the moments of the n-th "event". Exponential-Pareto distribution is the one of the inter-arrival times and Mixed Poisson-Pareto distribution describes the number of "events" up to time t. It turns out that:

- the Exponential - Pareto type is invariant with respect to integrated tail transformations. The distribution of the integrated tail increases the first parameter α to $\alpha + 1$;
- the mixed Poisson-Pareto type is invariant with respect to p-tinning. For all $p \in (0, 1)$ p-tinning procedure of MPPP is equivalent to change of the parameter δ to δp. From the theory of general MPP (See e.g. Grandell (1997)) it is well known that this is equivalent to $t \to tp$ change of time.

In the terms of applications, e.g. in insurance, Theorem 3 means that if we have i.i.d. claims arriving according to $MPPP(\alpha, \delta; t)$, $P(A) = p$ and consider only the claims satisfying A then the counting process of these claims is $MPPP(\alpha, \frac{\delta}{p}; t)$.

References

Grandell J. (1997) Mixed Poisson Processes, Chapman and Hall. London.

Greenwood, M., & Yule, G. U. (1920). An inquiry into the nature of frequency distributions representative of multiple happenings with particular reference to the occurrence of multiple attacks of disease or of repeated accidents. Journal of the Royal Statistical Society. Pages 255-279.

Jordanova, P., Dusek, J. & Stehlik, M. (2013) Modeling methane emission by the infinite moving average process. Chemometrics and Intelligent Laboratory Systems. Volume 122, pages 40-49.

Karlis D. & Xekalaki E., Mixed Poisson distributions. (2005). Int.Stat. Rev. Volume 73(1), pages 35-58.

Mecke, J.(1968) Eine charakteristiche Eigenschaft der doppelt stochastischen Poissonschen prozesse. Z. Wahrschein. und rev. Geb., Volume 11, Pages 74-81.

Mikosch, T. (2004) Non-Lide Insurance Mathematics: An Introduction with Stochastic Processes, Springer-Verlag, Berlin Heidelberg.

Milgram, M. S. (1985). The generalized integro-exponential function. Math. Comput. Volume 44, pages 443-458.

Olver, Fr. W. J., Lozier D. W., Boisvert R.F., & Clarc Ch. W. (2010). NIST Handbook on Mathematical Functions. Cambridge University Press.

Willmot G.E. (1993) On recursive evaluation of mixed Poisson probabilities and related quantities, Scand. Actuar. J. Volume 2, pages 114-133.